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A covariant approach to the quantization of a rigid body

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Abstract
This paper concerns the quantization of a rigid body in the framework of 'covariant quantum mechanics' on a curved spacetime with absolute time. We consider the configuration space of n classical particles as the n-fold product of the configuration space of one particle. Then, we impose a rigid constraint and the resulting space is dealt with as a configuration space of a single abstract 'particle'. This classical framework turns out to be suitable for the formulation of covariant quantum mechanics according to this scheme. Thus, we quantize such a 'particle' accordingly. This scheme can model, e.g., the quantum dynamics of extremely cold molecules. We provide a new mathematical interpretation of two-valued wavefunctions on $SO(3)$ in terms of single-valued sections of a new non-trivial quantum bundle. These results have clear analogies with spin.

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Introduction

A covariant formulation of classical and quantum mechanics on a curved spacetime with absolute time based on fibred manifolds, jets, nonlinear connections, cosymplectic forms and Frölicher smooth spaces has been proposed by Jadczyk and Modugno [26, 27] and further developed by several authors (see, for instance, [8, 28, 29, 43, 44, 49, 56]). We shall briefly call this approach 'covariant quantum mechanics'. It presents analogies with geometric quantization (see, for instance, [36, 50, 51, 57] and references therein), but several novelties as well. Indeed, it is formulated in a time-dependent framework, so that the covariance requirement is used as a leading guide. Moreover, in the flat case, it reproduces the standard quantum mechanics, hence it allows us to recover all classical examples (see [44] for a detailed comparison between the two approaches).

Here, we discuss an original geometric formulation of classical and quantum mechanics for a rigid body according to the general scheme of 'covariant quantum mechanics'. Our method, based on the classical multi-body and rigid model developed in detail in [45] and on the covariant quantum mechanics, seems to be a new approach, which is able to unify different cases on a clean mathematical scheme. It is clear that, from the physical viewpoint, our model can describe the dynamics of extremely cold molecules. In fact, the importance of vibrational modes in molecular quantum dynamics tends to decrease as the temperature becomes extremely low [23].

We start with a sketch of the essential features of the general 'covariant quantum mechanics' following [27, 28, 56]. The classical theory is based on a fibred manifold (spacetime) over time, equipped with a vertical Riemannian metric (space-like metric), a certain time and metric preserving linear connection (gravitational connection) and a closed 2-form (electromagnetic field). The above objects yield a cosymplectic 2-form on the first jet space of spacetime (phase space), in the sense of [13]. This 2-form controls the classical dynamics. The quantum theory is based on a Hermitian line bundle over spacetime (quantum bundle) equipped with a Hermitian 'universal' connection, whose curvature is proportional to the above classical cosymplectic 2-form. This quantum structure yields in a natural way a Lagrangian (hence the dynamics) and the quantum operators.

In view of the formulation of classical mechanics of a rigid body in the framework of the above scheme, we proceed in three steps [45].

Namely, we start with a flat spacetime for a pattern one-body mechanics.

Then, we consider an n-fold fibred product of the pattern structure as multi-spacetime for the n-body mechanics. A geometric 'product space' for n-body mechanics has been developed by several authors in different ways (see, for instance, [12, 14, 27, 40, 45]). In particular, our approach is close to that in [12] for the 'rotational' part of the rigid body dynamics, it can be easily compared with [41], and is close to [14] for the formulation of quantum structures. Moreover, we consider the subbundle of the multi-spacetime induced by a rigid constraint as configuration space for the rigid body mechanics.

The general machinery of covariant quantum mechanics can be applied to the above rigid body model. We discuss the existence and classification of the inequivalent quantum structures over the rigid configuration space. Quantum structures are pairs consisting of a Hermitian complex line bundle and a Hermitian connection, whose curvature is proportional to the cosymplectic 2-form. In the present framework it turns out that there are two possible quantum bundles: a trivial one and a non-trivial one. The transition functions of the non-trivial bundle are constant, hence both the trivial and the non-trivial bundles are endowed with a flat Hermitian connection. Such connections can be deformed by adding a dynamical term to produce two non-isomorphic quantum structures.

Then, we evaluate the classical 'translational' and 'rotational' observables of position, momenta ad energy and the corresponding quantum operators.

Finally, we explicitly compute the spectra of the rotational momentum and energy quantum operators for all quantum structures, in the cases of vanishing electromagnetic field ('free' rigid body), constant electric field (Stark effect), magnetic monopole field. The spectra are computed via the geometric techniques introduced by Casimir in his PhD thesis [9]. There are many computations in the literature for the spectra of a rigid body in some special electromagnetic fields (see, for instance, [1–4, 7, 9–11, 16, 19, 21, 23–25, 31, 34, 37–39, 42, 46–48, 52, 54, 55]).

Many of them can be recovered in our scheme by means of the quantum structure associated with the trivial quantum bundle.

The existence of a non-trivial quantum structure is an original result of our research. The non-trivial quantum bundle provides a clear mathematical setting and interpretation of the double-valued wavefunctions formalism. Classically [35] these functions were discarded because they were supposed to break the continuity of the quantum rotation operator, see also [11, 37, 42]. However, the requirement of single-valuedness of the wavefunctions was already studied by Pauli and Reiss in 1939, concluding that single-valuedness does not follow from basic quantum mechanical postulates and that certain kind of multivalued wavefunctions cannot be excluded. This conclusions have also been confirmed much more recently [3, 7]. Other authors have argued that it is only the probability density that must be single-valued, hence multi-valued wavefunctions, which from a mathematical point of view are described as sections of certain line bundles, must be accepted because their norm is single-valued; see [32] and the references therein. Therefore, in our approach, no continuity is broken if we allow the existence of a non-trivial quantum bundle.

But even before line bundles were invented, the most important contribution to this problem was given by Casimir in his PhD thesis [9]. On p 72 Casimir explains that the two-valuedness is due to the non-contractibility of the space of rotations:

> ... To a curve connecting ξ, η, ζ, χ with $-\xi, -\eta, -\zeta, -\chi$ there corresponds a closed motion that cannot be contracted; it may be changed into a rotation through 2π. Accordingly, we may say: the two-valuedness of the ξ_i (coordinates on $\mathbb{R}^4$ restricted to S^3), ..., the possibility of two-valued representations, are based on the kinematical fact that a 2π rotation cannot be contracted.

Thus, our non-trivial quantum bundle is a modern topological model implementing the above features: it appears exactly because the fundamental group of $SO(3)$ is $\mathbb{Z}_2$.

Our results imply the possibility of two quantum theories, one of which is formulated over a non-trivial bundle. One of the most important examples of application of the quantum rigid body is in molecular dynamics (see, for example, [23]). However, as far as we know, in all cases in which the spectrum of molecules has half-integer eigenvalues, this value can be attributed to the spin of constituent particles while in our model we assumed the n-bodies to be scalar particles. This is a very interesting phenomenon that would deserve a deeper physical analysis. Our results show that the theoretical possibility of a rigid body with half-integer angular momentum exists; on the other hand, there could be a superselection rule by which nature always chooses the trivial bundle.

We stress that the occurrence of topological phenomena in quantum mechanics has been observed in some cases (e.g., the Aharonov–Bohm effect; see [56] for a description in covariant quantum mechanics). In other cases, however, it has only been predicted (quantization of charge for a monopole). So we cannot *a priori* exclude the possibility that the quantum theory on the non-trivial bundle could play a physical role. However, regardless of this problem, our quantum theory of a rigid body could be taken, in a certain sense, as a basis for a kind of 'semi-classical model of spin', by taking into account the scheme of covariant quantum mechanics for a spin particle [8]. Indeed, some authors have considered such a possibility in different non-covariant frameworks (see, for instance [3, 7, 19, 22]).

Finally, we invite the reader to take a look at the conclusions (section 5) for possible directions of research on the topic.

We assume manifolds and maps to be C^∞. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are manifolds, then the sheaf of local smooth maps $\boldsymbol{M} \to \boldsymbol{N}$ is denoted by $\mathrm{map}(\boldsymbol{M}, \boldsymbol{N})$.

1. Covariant quantum mechanics

We start with a brief sketch of the basic notions of 'covariant quantum mechanics', paying attention just to the facts that are strictly needed in the present paper. We follow [27, 28, 30, 43, 49, 56]. For further details and discussions the reader should refer to the above literature and references therein.

In order to make classical and quantum mechanics explicitly independent from scales, we introduce the 'spaces of scales'. Roughly speaking, a space of scales has the algebraic structure of $\mathbb{R}^+$ but has no distinguished 'basis'. The basic objects of our theory (metric, electromagnetic field, etc) will be valued into *scaled* vector bundles, that is into vector bundles twisted with spaces of scales. We shall use rational tensor powers of spaces of scales. In this way, each tensor field carries explicit information on its 'scale dimension'.

Actually, we assume the following basic spaces of scales: the space of *time intervals* $\mathbb{T}$, the space of *lengths* $\mathbb{L}$, the space of *masses* $\mathbb{M}$.

We assume the *Planck's constant* $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$. Moreover, a *particle* will be assumed to have a *mass* $m \in \mathbb{M}$ and a *charge* $q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

1.1. Classical scheme

We assume

– the *time* to be an affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathbb{R}$,
– the *spacetime* to be an oriented manifold $\boldsymbol{E}$ of dimension $1 + 3$,
– the *time fibring* to be a fibring (i.e., a surjective submersion) $t : \boldsymbol{E} \to \boldsymbol{T}$,
– the *space-like metric* to be a scaled vertical Riemannian metric

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes S^2 V^* \boldsymbol{E},$$

– the *gravitational connection* to be a linear connection of spacetime

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

such that $\nabla[K^\natural]\mathrm{d}t = 0$ and $\nabla[K^\natural]g = 0$, and whose curvature $R[K^\natural]$ is 'vertically symmetric',
– the *electromagnetic field* to be a closed scaled 2-form

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E}.$$

The space-like orientation and the metric g yield the space-like scaled volume form η and its dual $\bar{\eta}$.

With reference to a given particle with mass m and charge q, it is convenient to consider the *rescaled* sections $G := \frac{m}{\hbar} g$ and $\frac{q}{\hbar} F$.

We shall refer to fibred charts (x^0, x^i) of $\boldsymbol{E}$, where x^0 is adapted to the affine structure of $\boldsymbol{T}$ and to a time scale $u_0 \in \mathbb{T}$. Latin indices $i, j, \ldots$ and Greek indices $\lambda, \mu, \ldots$ will label space-like and spacetime coordinates, respectively. For short, we shall denote the induced dual bases of vector fields and forms by ∂_λ and d^λ. The vertical restriction of forms will be denoted by the check "$^\vee$".

We have the coordinate expression $G = G^0_{ij} u_0 \otimes \check{d}^i \otimes \check{d}^j$. The coordinate expression of the condition of vertical symmetry of $R[K^\natural]$ is $R^\natural{}_{i\lambda j\mu} = R^\natural{}_{j\mu i\lambda}$.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$.

We assume the first jet space of motions $J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ as *phase space* for classical mechanics of a spinless particle; the first jet prolongation $j_1 s$ of a motion s is said to be its *velocity*. We denote by $\left(x^\lambda, x^i_0\right)$ the chart induced on $J_1\boldsymbol{E}$. We shall use the natural complementary maps $д : J_1\boldsymbol{E} \times \bar{\mathbb{T}} \to T\boldsymbol{E}$ and $\theta : J_1\boldsymbol{E} \times_{\boldsymbol{E}} T\boldsymbol{E} \to V\boldsymbol{E}$, with coordinate expressions $д = u^0 \otimes (\partial_0 + x^i_0 \partial_i)$ and $\theta = (d^i - x^i_0 d^0) \otimes \partial_i$. We set $\theta^i \equiv d^i - x^i_0 d^0$.

An *observer* is defined to be a (local) section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$.

An observer o is said to be *rigid* if the Lie derivative $L[o]g$ vanishes.

Let us consider an observer o.

A chart (x^0, x^i) is said to be *adapted* to o if $o^i_0 \equiv x^i_0 \circ o = 0$. We obtain the maps $\nu[o] : T\boldsymbol{E} \to V\boldsymbol{E} : X \to X - o \lrcorner \mathrm{d}t(X)$ and $\nabla[o] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E} : e_1 - o(e)$. We define the *observed component* of a vector $v \in T\boldsymbol{E}$, to be the space-like vector $\vec{v}[o] := \nu[o](v)$. Accordingly, if s is a motion, then we define the *observed velocity* to be the section $\nabla[o] \circ j_1 s : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E}$.

We define the *observed kinetic energy* and *momentum*, respectively, as the maps

$$\mathcal{K}[o] := \tfrac{1}{2}\, G(\nabla[o], \nabla[o]) : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \quad \text{and} \quad \mathcal{Q}[o] := \theta^* \circ G^\flat(\nabla[o]) : J_1\boldsymbol{E} \to T^*\boldsymbol{E},$$

with coordinate expressions $\mathcal{K}[o] = \frac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 d^0$ and $\mathcal{Q}[o] = G^0_{ij}\, x^j_0 \theta^i$.

We have the observed splitting

$$F = -2\mathrm{d}t \wedge E[o] + 2\nu^*[o](\mathrm{i}(\vec{B})\eta), \tag{1}$$

where the *magnetic field* and the *observed electric field*, are respectively defined as $\vec{B} := \frac{1}{2}\,\mathrm{i}(F^\vee)\bar{\eta}$ and $\vec{E}[o] := -g^\sharp(E[o]) = -g^\sharp((o \lrcorner F)^\vee)$.

The linear connection $K^\natural$ yields an affine connection $\Gamma^\natural$ of the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, with coordinate expression $\Gamma^\natural{}_{\lambda}{}^{i}{}_{0}{}^{0}{}_{\mu} = K^\natural{}_{\lambda}{}^{i}{}_{\mu}$, and the nonlinear connection $\gamma^\natural := д \lrcorner \Gamma^\natural : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$ of the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$, with coordinate expression $\gamma^\natural = u^0 \otimes (\partial_0 + x^i_0 \partial_i + \gamma^\natural{}^i_{00} \partial^0_i)$, where $\gamma^\natural{}^i_{00} := K^\natural{}_h{}^i{}_k x^h_0 x^k_0 + 2K^\natural{}_h{}^i{}_0 x^h_0 + K^\natural{}_0{}^i{}_0$. Moreover, $\Gamma^\natural$ yields the 2-form $\Omega^\natural := G(\nu[\Gamma^\natural] \wedge \theta) : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$. We have the coordinate expression $\Omega^\natural = G^0_{ij}(d^i_0 - \gamma^\natural{}^i_{00} d^0 - \Gamma^\natural{}_h{}^i_0 \theta^h) \wedge \theta^j$, where $\Gamma^\natural{}_h{}^i_0 \equiv \Gamma^\natural{}_h{}^i_0{}^0_k x^k_0 + \Gamma^\natural{}_h{}^i_0{}^0_0$. The 2-form $\Omega^\natural$ turns out to be closed, in virtue of the assumed symmetry of $R[K^\natural]$, and *non-degenerate* as $\mathrm{d}t \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural$ is a scaled volume form of $J_1\boldsymbol{E}$. Thus, $\Omega^\natural$ turns out to be a *cosymplectic form*.

There is a natural geometric way to 'merge' the gravitational and electromagnetic objects into *joined* objects, in such a way that all mutual relations holding for gravitational objects are preserved for joined objects. In particular, we deal with the joined 2-form $\Omega := \Omega^\natural + \frac{1}{2}\frac{q}{\hbar} F$ and the joined connection $\gamma = \gamma^\natural + \gamma^e$, where γ^e turns out to be the Lorentz force $\gamma^e = -G^\sharp(д \lrcorner F)^\vee$.

We obtain $\mathrm{d}\Omega = 0$ and $\mathrm{d}t \wedge \Omega \wedge \Omega \wedge \Omega = \mathrm{d}t \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural$. Thus, also Ω turns out to be a *cosymplectic form*. It rules the classical dynamics in the following way. The closed form Ω admits local 'horizontal' potentials of the type $A^\uparrow : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, whose coordinate expression is of the type $A^\uparrow = -\big(\frac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0\big) d^0 + \big(G^0_{ij}\, x^j_0 + A_i\big) d^i$, that is, for each observer o, of the type $A^\uparrow = -\mathcal{K}[o] + \mathcal{Q}[o] + A[o]$, where $A[o] := o^* A^\uparrow : \boldsymbol{E} \to T^*\boldsymbol{E}$.

We define the (local) *Lagrangian* $\mathcal{L}[A^\uparrow] := д \lrcorner A^\uparrow : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ and the (local) *momentum* $\mathcal{P}[A^\uparrow] := \theta^* V_{\boldsymbol{E}} \mathcal{L}[A^\uparrow] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, with expressions $\mathcal{L}[A^\uparrow] = \big(\frac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_i x^i_0 + A_0\big) d^0$ and $\mathcal{P}[A^\uparrow] = \big(G^0_{ij}\, x^j_0 + A_i\big)\theta^i$. Indeed, the Poincaré–Cartan form $\Theta = \mathcal{L}[A^\uparrow] + \mathcal{P}[A^\uparrow]$ associated with $\mathcal{L}[A^\uparrow]$ turns out to be just $A^\uparrow$.

Moreover, given an observer o, we define the (observed) *Hamiltonian* $\mathcal{H}[A^\uparrow, o] := -o \lrcorner A^\uparrow : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ and the (observed) *momentum* $\mathcal{P}[A^\uparrow, o] := \nu[o] \lrcorner A^\uparrow : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, with coordinate expressions $\mathcal{H}[A^\uparrow, o] = \big(\frac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0\big) d^0$ and $\mathcal{P}[A^\uparrow, o] = \big(G^0_{ij}\, x^j_0 + A_i\big) d^i$, in adapted coordinates. We also obtain the scaled function $\|\mathcal{P}[A^\uparrow, o]\|^2$, with a coordinate expression $\|\mathcal{P}[A^\uparrow, o]\|^2_0 = G^0_{ij} x^i_0 x^j_0 + 2A_i x^i_0 + G^{ij}_0 A_i A_j$.

The Euler–Lagrange equation, in the unknown motion s, associated with the (local) Lagrangians $\mathcal{L}[A^\uparrow]$ coincides with the global equation $\nabla[\gamma] j_1 s = 0$, that is $\nabla[\gamma^\natural] j_1 s =$

$\gamma^e \circ j_1 s$. This equation is just the generalized Newton's equation of motion for a charged particle in the given gravitational and electromagnetic fields. Note that the above equation reduces to the standard Newton's equation in an inertial frame, but it can be written in an arbitrary (non-inertial, accelerated) coordinate system. See [27, 49] for its coordinate expression. We assume this equation to be our classical equation of motion.

1.2. Quantum scheme

A *quantum bundle* is defined to be a complex line bundle $\boldsymbol{Q} \to \boldsymbol{E}$, equipped with a Hermitian metric h with values in $\mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}$. A *quantum section* $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ describes a quantum particle.

A local section $\mathrm{b} : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q}$, such that $\mathrm{h}(\mathrm{b}, \mathrm{b}) = \eta$, is a local basis. We denote the local complex dual basis of b by $z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$. If Ψ is a quantum section, then we write locally $\Psi = \psi \mathrm{b}$, where $\psi := z \circ \Psi : \boldsymbol{E} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$.

The *Liouville vector field* is defined to be the vector field $\mathbb{I} : \boldsymbol{Q} \to V\boldsymbol{Q} : q \mapsto (q, q)$.

Lets us consider the *phase quantum bundle* $\boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \times_{\boldsymbol{E}} \boldsymbol{Q} \to J_1\boldsymbol{E}$.

If $\{\text{Ч}[o]\}$ is a family of Hermitian connections of $\boldsymbol{Q} \to \boldsymbol{E}$ parametrized by the observers o, then there is a unique Hermitian connection $\text{Ч}^\uparrow$ of $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$, such that $\text{Ч}[o] = o^*\text{Ч}^\uparrow$, for each observer o. This connection is called *universal* and is locally of the type $\text{Ч}^\uparrow = \chi^\uparrow[\mathrm{b}] + \mathrm{i}\, A^\uparrow[\mathrm{b}] \otimes \mathbb{I}^\uparrow$, where $\chi^\uparrow[\mathrm{b}]$ is the flat connection induced by a local quantum basis b and $A^\uparrow[\mathrm{b}]$ is a local horizontal 1-form of $J_1\boldsymbol{E}$. The map $\{\text{Ч}[o]\} \mapsto \text{Ч}^\uparrow$ is a bijection.

We define a *phase quantum connection* to be a connection $\text{Ч}^\uparrow$ of the phase quantum bundle, which is Hermitian, universal and whose curvature is $R[\text{Ч}^\uparrow] = -2\mathrm{i}\,\Omega \otimes \mathbb{I}^\uparrow$. A phase quantum connection $\text{Ч}^\uparrow$ is locally of the type $\text{Ч}^\uparrow = \chi^\uparrow[\mathrm{b}] + \mathrm{i}\, A^\uparrow[\mathrm{b}] \otimes \mathbb{I}^\uparrow$, where $A^\uparrow[\mathrm{b}]$ is a local horizontal potential for Ω. We remark that the equation $\mathrm{d}\Omega = 0$ turns out to be just the Bianchi identity for a phase quantum connection $\text{Ч}^\uparrow$.

A pair $(\boldsymbol{Q}, \text{Ч}^\uparrow)$ is said to be a *quantum structure*.

Two quantum bundles $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ on $\boldsymbol{E}$ are said to be *equivalent* if there exists an isomorphism of Hermitian line bundles $f : \boldsymbol{Q}_1 \to \boldsymbol{Q}_2$ over $\boldsymbol{E}$ (the existence of such an f is equivalent to the existence of an isomorphism of line bundles). Two quantum structures $(\boldsymbol{Q}_1, \text{Ч}^\uparrow{}_1)$ and $(\boldsymbol{Q}_2, \text{Ч}^\uparrow{}_2)$ are said to be *equivalent* if there exists an equivalence $f : \boldsymbol{Q}_1 \to \boldsymbol{Q}_2$ which maps $\text{Ч}^\uparrow{}_1$ into $\text{Ч}^\uparrow{}_2$. A quantum bundle is said to be *admissible* if it admits a phase quantum connection. Actually, the following results hold.

Let us consider the cohomology $H^*(\boldsymbol{E}, X)$ with values in $X = \mathbb{R}$, or $X = \mathbb{Z}$, the inclusion $i : \mathbb{Z} \to \mathbb{R}$ and the induced group morphism $i^* : H^*(\boldsymbol{E}, \mathbb{Z}) \to H^*(\boldsymbol{E}, \mathbb{R})$.

The difference of two local horizontal potentials for Ω turns out to be a closed spacetime form. Therefore the de Rham class $[\Omega]_R$ yields a cohomology class $[\Omega] \in H^2(\boldsymbol{E}, \mathbb{R})$.

Proposition 1.1 [56]. *We have the following classification results.*

(1) The equivalence classes of complex line bundles on $\boldsymbol{E}$ are in bijection with $H^2(\boldsymbol{E}, \mathbb{Z})$.
(2) There exists a quantum structure on $\boldsymbol{E}$, if and only if

$$[\Omega] \in i^2\big(H^2(\boldsymbol{E}, \mathbb{Z})\big) \subset H^2(\boldsymbol{E}, \mathbb{R}) \simeq H^2(J_1\boldsymbol{E}, \mathbb{R}).$$

(3) Equivalence classes of quantum structures are in bijection with the set

$$(i^2)^{-1}([\Omega]) \times H^1(\boldsymbol{E}, \mathbb{R})/H^1(\boldsymbol{E}, \mathbb{Z}).$$

More precisely, the first factor parametrizes admissible quantum bundles and the second factor parametrizes phase quantum connections.

The quantum theory is based on the only assumption of a quantum structure, supposing that the background spacetime admits one. Namely, we assume a quantum bundle $\boldsymbol{Q}$ equipped with a phase quantum connection $\Psi^\uparrow$. All further quantum objects will be derived from the above quantum structure by natural procedures.

We have been forced to assume that $\Psi^\uparrow$ lives on the phase quantum bundle $\boldsymbol{Q}^\uparrow$ because of the required link with the 2-form Ω. On the other hand, in order to accomplish the covariance of the theory, we wish to derive from $\Psi^\uparrow$ new quantum objects, which are observer independent, hence living on the quantum bundle. For this purpose, we follow a successful projectability procedure: if $\boldsymbol{V}^\uparrow \to J_1\boldsymbol{E}$ is a vector bundle which projects on a vector bundle $\boldsymbol{V} \to \boldsymbol{E}$, then we look for sections $\sigma^\uparrow : J_1\boldsymbol{E} \to \boldsymbol{V}^\uparrow$ which are projectable on sections $\sigma : \boldsymbol{E} \to \boldsymbol{V}$ and take these σ as candidates to represent quantum objects.

The quantum connection allows us to perform covariant derivatives of sections of $\boldsymbol{Q}$ (via pullback). Then, given an observer o, the *observed quantum connection* $\Psi[o] := o^*\Psi^\uparrow$ yields, for each section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$, the *observed quantum differential* and the *observed quantum Laplacian*, with coordinate expressions

$$\overset{o}{\nabla}_\lambda \psi = (\partial_\lambda - \mathrm{i}\, A_\lambda)\psi,$$
$$\overset{o}{\Delta}_0 \psi = \left(G_0^{hk}(\partial_h - \mathrm{i}\, A_h)(\partial_k - \mathrm{i}\, A_k) + \frac{\partial_h\big(G_0^{hk}\sqrt{|g|}\big)}{\sqrt{|g|}}(\partial_k - \mathrm{i}\, A_k) \right)\psi.$$

We can prove that all first-order covariant quantum Lagrangians [28] are of the type (we recall that $m/\hbar$ has been incorporated into G and $A[o]$)

$$\begin{aligned}\mathrm{L}[\Psi] = \tfrac{1}{2}\big(&\mathrm{i}\,(\bar\psi\partial_0\psi - \psi\partial_0\bar\psi) + 2A_0\bar\psi\psi \\ &- G_0^{ij}(\partial_i\bar\psi\partial_j\psi + A_iA_j\bar\psi\psi) - \mathrm{i}\,A_0^i(\bar\psi\partial_i\psi - \psi\partial_i\bar\psi) + k\rho_0\bar\psi\psi\big)\sqrt{|g|}d^0\wedge d^1\wedge d^2\wedge d^3,\end{aligned}$$

where $\rho_0 = G_0^{ij}R_{hi}{}^h{}_j$ is the scalar curvature of the spacetime connection K and $k \in \mathbb{R}$ is an arbitrary parameter (which cannot be determined by covariance arguments).

By a standard procedure, these Lagrangians yield the quantum momentum, the Euler–Lagrange operator (generalized *Schrödinger operator*) and a conserved form (*probability current*). We assume the quantum sections Ψ to fulfil the generalized Schrödinger equation with a coordinate expression (we recall that $m/\hbar$ has been incorporated into G and $A[o]$)

$$\mathrm{S}_0\psi = \left(\overset{o}{\nabla}_0 + \frac{1}{2}\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{1}{2}\,\mathrm{i}\,(\overset{o}{\Delta}_0 + k\rho_0)\right)\psi = 0.$$

Next, we sketch the formulation of quantum operators.

We can exhibit a distinguished Lie algebra $\mathrm{spec}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{map}(\mathrm{J}_1\boldsymbol{E}, \mathbb{R})$ of functions, called *special phase functions*, of the type $f = f^0\frac{1}{2}G^0_{ij}x_0^ix_0^j + f^iG^0_{ij}x_0^j + \breve{f}$, where $f^\lambda, \breve{f} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$. Among special phase functions we have $x^\lambda, \mathcal{P}_j, \mathcal{H}_0$ and $\|\mathcal{P}\|_0^2$. The bracket of this algebra is defined in terms of the Poisson bracket and γ.

Then, by classifying the vector fields on $\boldsymbol{Q}$ which preserve the Hermitian metric and are projectable on $\boldsymbol{E}$ and on $\boldsymbol{T}$, we see that they constitute a Lie algebra, which is naturally isomorphic to the Lie algebra of special phase functions. These vector fields can be regarded as *pre-quantum operators* $Z[f]$ acting on quantum sections.

The *sectional quantum bundle* is defined to be the bundle $\hat{\boldsymbol{Q}} \to \boldsymbol{T}$, whose fibres $\hat{\boldsymbol{Q}}_\tau$, with $\tau \in \boldsymbol{T}$, are constituted by smooth quantum sections, at the time τ, with compact support. This infinite-dimensional complex vector bundle turns out to be F-smooth in the sense of Frölicher

[17] and inherits a pre-Hilbert structure via integration over the fibres. A Hilbert bundle can be obtained by completion, and we can prove that the Schrödinger operator S can be naturally regarded as a linear connection of $\hat{\boldsymbol{Q}} \to \boldsymbol{T}$.

Eventually, a natural procedure associates with every special phase function f a symmetric *quantum operator* $\hat{f} : \hat{\boldsymbol{Q}} \to \hat{\boldsymbol{Q}}$, fibred over $\boldsymbol{T}$, defined as a linear combination of the corresponding pre-quantum operator $Z[f]$ and of the operator $f^0 \mathrm{S}_0$. We obtain the coordinate expression (we recall that $m/\hbar$ has been incorporated into G and $A[o]$)

$$\hat{f}\psi = \left(\check{f} - \mathrm{i}\, f^h(\partial_h - \mathrm{i}\, A_h) - \mathrm{i}\, \frac{1}{2}\, \frac{\partial_h(f^h\sqrt{|g|})}{\sqrt{|g|}} - \frac{1}{2}\, f^0(\overset{o}{\Delta}_0 + k\rho_0)\right)\psi.$$

For example, we have

$$\widehat{x^0}\psi = x^0\psi, \qquad \widehat{x^i}\psi = x^i\psi,$$
$$\widehat{\mathcal{P}_j}\psi = -\mathrm{i}\left(\partial_j + \frac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\right)\psi, \qquad \widehat{\mathcal{H}_0}\psi = \left(-\frac{1}{2}\,(\overset{o}{\Delta}_0 + k\rho_0) - A_0\right)\psi,$$
$$\widehat{\|\mathcal{P}\|_0^2}\psi = \left(-G_0^{ij}A_iA_j - \mathrm{i}\,\frac{\partial_h(A_0^h\sqrt{|g|})}{\sqrt{|g|}} - 2\mathrm{i}\, A_0^h\partial_h - (\overset{o}{\Delta}_0 + k\rho_0)\right)\psi.$$

2. Rigid body classical mechanics

The configuration space of the classical rigid body is formulated in three steps according to [45]

- we start with a flat *pattern spacetime* of dimension 1 + 3 for the formulation of one-body classical and quantum mechanics;
- then, we consider the *n*-fold fibred product of the pattern spacetime, as the framework for *n*-body classical mechanics. Note that the metric and gravitational electromagnetic fields naturally equip this multi-spacetime with analogous 'multi' fields. The multi-fields involve suitable weights related to the masses and charges of the particles.
- finally, we consider the rigid constrained fibred submanifold of the above *n*-fold fibred product along with the induced structures, as the framework for classical rigid-body.

As a result, our system of particles is described as a single 'particle' moving in a higher dimensional spacetime equipped with suitable fields which fulfil the same properties postulated for the standard spacetime.

2.1. One-body mechanics

Let us consider a system of one particle, with mass m and charge q.

We assume as *pattern spacetime* a (1+3)-dimensional affine space $\boldsymbol{E}$, associated with the vector space $\bar{\boldsymbol{E}}$ and equipped with an affine map $t : \boldsymbol{E} \to \boldsymbol{T}$ as time map.

From the above affine structure follow some immediate consequences.

The map $Dt : \bar{\boldsymbol{E}} \to \bar{\mathbb{T}}$ yields the three-dimensional vector subspace $\boldsymbol{S} := Dt^{-1}(0) \subset \bar{\boldsymbol{E}}$ and the three-dimensional affine subspace $\boldsymbol{U} := (\mathrm{id}[\mathbb{T}^*] \otimes Dt)^{-1}(1) \subset \mathbb{T}^* \otimes \bar{\boldsymbol{E}}$, which is associated with the vector space $\mathbb{T}^* \otimes \boldsymbol{S}$. Thus, $t : \boldsymbol{E} \to \boldsymbol{T}$ turns out to be a principal bundle associated with the Abelian group $\boldsymbol{S}$. Moreover, we have the natural isomorphisms $T\boldsymbol{E} \simeq \boldsymbol{E} \times \bar{\boldsymbol{E}}$, $V\boldsymbol{E} \simeq \boldsymbol{E} \times \boldsymbol{S}$ and $J_1\boldsymbol{E} \simeq \boldsymbol{E} \times \boldsymbol{U}$.

We assume a Euclidean metric $g \in \mathbb{L}^2 \otimes (\boldsymbol{S}^* \otimes \boldsymbol{S}^*)$ as a space-like metric. Moreover, we assume the connection $K^\natural$ induced by the affine structure as the gravitational connection. Furthermore, we assume an electromagnetic field F. Thus, we obtain $\mathrm{d}\Omega^\natural = 0$ and $dF = 0$.

Moreover, because of the affine structure of spacetime, $\Omega^\natural$ and F turn out to be globally exact. We denote global potentials for $\Omega^\natural$ and F by $A^{\uparrow\natural}$ and $A^{\mathfrak{e}}$.

A motion s and an observer o are said to be *inertial* if they are affine maps. Any inertial observer yields a splitting of the type $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}[o]$, where $\boldsymbol{P}[o]$ is an affine space associated with $\boldsymbol{S}$. Any inertial motion yields an inertial observer o and an affine isomorphism $\boldsymbol{P}[o] \simeq \boldsymbol{S}$. For each inertial observer o, we obtain the splitting $A^{\uparrow\natural} = -\mathcal{K}[o] + \mathcal{Q}[o] + A^\natural[o]$, where $A^\natural[o] \in \bar{\boldsymbol{E}}^*$ is a 1-form.

2.2. Multi-body mechanics

Let us consider a system of n particles, with $n \geqslant 2$, and with masses $m_1, \dots, m_n$ and charges $q_1, \dots, q_n$. We define the *total mass* $m := \sum_i m_i$, the *ith weight* $\mu_i := m_i/m \in \mathbb{R}^+$ and the *total charge* $q := \sum_i q_i$. Of course, we have $\sum_i \mu_i = 1$.

In order to label the different particles of the system, we introduce n identical copies of the pattern objects $\boldsymbol{E}_i \equiv \boldsymbol{E}$, $\boldsymbol{S}_i \equiv \boldsymbol{S}$, $\boldsymbol{U}_i \equiv \boldsymbol{U}$, $g_i \equiv g$, $F_i \equiv F$, for $i = 1, \dots, n$. We assume the fibred product over $\boldsymbol{T}$,

$$\boldsymbol{E}_{\text{mul}} := \boldsymbol{E}_1 \underset{\boldsymbol{T}}{\times} \cdots \underset{\boldsymbol{T}}{\times} \boldsymbol{E}_n,$$

as *multi-spacetime*, equipped with the associated projection $t_{\text{mul}} : \boldsymbol{E}_{\text{mul}} \to \boldsymbol{T}$. The affine multi-spacetime $\boldsymbol{E}_{\text{mul}}$ is associated with the multi-vector space $\bar{\boldsymbol{E}}_{\text{mul}} = \bar{\boldsymbol{E}}_1 \times_{\bar{\mathbb{T}}} \ldots \times_{\bar{\mathbb{T}}} \bar{\boldsymbol{E}}_n$, which turns out to be a principal bundle $Dt_{\text{mul}} : \bar{\boldsymbol{E}}_{\text{mul}} \to \bar{\mathbb{T}}$, associated with the vector space $\boldsymbol{S}_{\text{mul}} := \boldsymbol{S}_1 \times \cdots \times \boldsymbol{S}_n$.

Each observer o yields the *multi-observer* $o_{\text{mul}} := (o \times \cdots \times o)$.

Moreover, we assume the Euclidean metrics

$$g_{\text{mul}} := (\mu_1 g_1 \times \cdots \times \mu_n g_n) \qquad \text{and} \qquad G_{\text{mul}} := \frac{m}{\hbar}\, g_{\text{mul}} := \left(\frac{m_1}{\hbar}\, g_1 \times \cdots \times \frac{m_n}{\hbar}\, g_n\right)$$

as *multi-spacelike metric* and *rescaled multi-spacelike metric*, the affine connection and the 2-form

$$K^\natural{}_{\text{mul}} := K^\natural{}_1 \times \cdots \times K^\natural{}_n \qquad \text{and} \qquad \mathcal{F}_{\text{mul}} := \left(\frac{q_1}{m}\, F_1 \times \cdots \times \frac{q_n}{m}\, F_n\right)$$

as *multi-gravitational connection* and *rescaled multi-electromagnetic field*.

We define the *multi-magnetic field* and the *observed multi-electric field*, respectively, as $\vec{\mathcal{B}}_{\text{mul}} := \frac{1}{2}\, \mathrm{i}(\mathcal{F}^\vee)\bar{\eta}_{\text{mul}}$ and $\vec{\mathcal{E}}_{\text{mul}}[o_{\text{mul}}] := -g^\sharp_{\text{mul}}(\mathcal{E}_{\text{mul}}[o_{\text{mul}}])$, where $\mathcal{E}_{\text{mul}}[o_{\text{mul}}] = (o_{\text{mul}} \lrcorner \mathcal{F}_{\text{mul}})^\vee$. Then, we obtain the observed splitting $\mathcal{F}_{\text{mul}} = -2\mathrm{d}t_{\text{mul}} \wedge \mathcal{E}_{\text{mul}}[o_{\text{mul}}] + 2\nu^*[o_{\text{mul}}](\mathrm{i}(\vec{\mathcal{B}})\eta_{\text{mul}})$.

The above multi-spacetime and multi-fields yield further several multi-objects analogously to the case of the pattern spacetime and pattern fields. In particular, we obtain $\mathrm{d}\Omega^\natural{}_{\text{mul}} = 0$ and $\mathrm{d}\mathcal{F}_{\text{mul}} = 0$. Moreover, Ω_{mul} and $\mathcal{F}_{\text{mul}}$ are globally exact.

Due to the affine structure and the weights of masses, the multi-spacetime is equipped with another important splitting, which is related to the centre of mass. Namely, the multi-spacetime splits naturally into the product of the $(3+1)$-dimensional affine subspace of centre of mass and the $(3n-3)$-dimensional vector space of distances relative to the centre of mass. This splitting will affect all geometric, kinematical and dynamical structures, including the equation of motion.

Let us consider a copy $\boldsymbol{E}_{\text{cen}} := \boldsymbol{E}$ of the pattern spacetime, referred to as the *space of centre of mass*. We define the affine fibred projection of the *centre of mass*

$$\pi_{\text{cen}} : \boldsymbol{E}_{\text{mul}} \to \boldsymbol{E}_{\text{cen}} : e_{\text{mul}} \equiv (e_1, \dots, e_n) \mapsto e_{\text{cen}}, \qquad \text{with} \qquad \sum_i \mu_i (e_i - e_{\text{cen}}) \equiv 0.$$

Let us consider the three-dimensional diagonal affine subspace $i_{\text{dia}} : \boldsymbol{E}_{\text{dia}} \hookrightarrow \boldsymbol{E}_{\text{mul}}$. Clearly, the restriction of π_{cen} to $\boldsymbol{E}_{\text{dia}}$ yields an affine fibred isomorphism $\boldsymbol{E}_{\text{dia}} \to \boldsymbol{E}_{\text{cen}}$. We shall often identify these two spaces via the above isomorphism and write $i_{\mathrm{cen}} : \boldsymbol{E}_{\mathrm{cen}} \hookrightarrow \boldsymbol{E}_{\mathrm{mul}}$. Moreover, we define the *centre-of-mass space* and the *relative space* to be, respectively, the three-dimensional and the $(3n-3)$-dimensional vector subspaces of $\boldsymbol{S}_{\mathrm{mul}}$:

$$\boldsymbol{S}_{\mathrm{cen}} := \{v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}} \mid v_1 = \ldots = v_n\}, \qquad \boldsymbol{S}_{\mathrm{rel}} := \left\{ v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}} \,\middle|\, \sum_i \mu_i v_i = 0 \right\}. \tag{2}$$

We set $\boldsymbol{E}_{\mathrm{rel}} := \boldsymbol{T} \times \boldsymbol{S}_{\mathrm{rel}}$. We obtain the affine fibred splitting over $\boldsymbol{T}$,

$$\boldsymbol{E}_{\mathrm{mul}} \to \boldsymbol{E}_{\mathrm{cen}} \underset{\boldsymbol{T}}{\times} \boldsymbol{E}_{\mathrm{rel}} = \boldsymbol{E}_{\mathrm{cen}} \times \boldsymbol{S}_{\mathrm{rel}} :$$

$$e_{\mathrm{mul}} \mapsto (e_{\mathrm{cen}}, v_{\mathrm{rel}}) := (\pi_{\mathrm{cen}}(e_{\mathrm{mul}}), e_{\mathrm{mul}} - i_{\mathrm{dia}}(e_{\mathrm{cen}})).$$

We stress that there is no natural inclusion $\boldsymbol{S}_{\mathrm{rel}} \to \boldsymbol{E}_{\mathrm{mul}}$. We have the further splittings

$$\bar{\boldsymbol{E}}_{\mathrm{mul}} \to \bar{\boldsymbol{E}}_{\mathrm{cen}} \times \boldsymbol{S}_{\mathrm{rel}} :$$

$$(v_1, \ldots, v_n) \mapsto \left(\sum_i \mu_i v_i, \left(v_1 - \sum_i \mu_i v_i, \ldots, v_n - \sum_i \mu_i v_i \right) \right),$$

$$\bar{\boldsymbol{E}}^*_{\mathrm{mul}} \to \bar{\boldsymbol{E}}^*_{\mathrm{cen}} \times \boldsymbol{S}^*_{\mathrm{rel}} :$$

$$(\alpha_1, \ldots, \alpha_n) \mapsto \left(\sum_i \alpha_i, \left(\alpha_1 - \mu_1 \left(\sum_i \alpha_i \right), \ldots, \alpha_n - \mu_n \left(\sum_i \alpha_i \right) \right) \right).$$

They turn out to be affine fibred splittings over $\bar{\boldsymbol{T}}$ orthogonal with respect to G_{mul}.

The multi-metric g_{mul} splits into the product of a metric $g_{\mathrm{dia}} \simeq g_{\mathrm{cen}}$ of $\boldsymbol{E}_{\mathrm{dia}} \simeq \boldsymbol{E}_{\mathrm{cen}}$ and a metric g_{rel} of $\boldsymbol{S}_{\mathrm{rel}}$. We observe that $g_{\mathrm{cen}} = g$, in virtue of the equality $\sum_i \mu_i = 1$. Therefore, the multi-metric G_{mul} splits into the product of the metric $G_{\mathrm{dia}} \simeq G_{\mathrm{cen}} = \frac{m}{\hbar} g$ of $\boldsymbol{E}_{\mathrm{dia}} \simeq \boldsymbol{E}_{\mathrm{cen}}$ and the metric $G_{\mathrm{rel}} = \frac{m}{\hbar} g_{\mathrm{rel}}$ of $\boldsymbol{E}_{\mathrm{rel}}$.

The gravitational connection $K^{\natural}{}_{\mathrm{mul}}$ of the multi-spacetime $\boldsymbol{E}_{\mathrm{mul}}$ splits into the product of a gravitational connection $K^{\natural}{}_{\mathrm{cen}}$ of $\boldsymbol{E}_{\mathrm{cen}}$ and of a gravitational connection $K^{\natural}{}_{\mathrm{rel}}$ of $\boldsymbol{S}_{\mathrm{rel}}$. The connections $K^{\natural}{}_{\mathrm{cen}}$ and $K^{\natural}{}_{\mathrm{rel}}$ coincide with the connections induced by the affine structures of the corresponding spaces (because affine isomorphisms between affine spaces preserve the connections induced by the affine structures). Moreover, the connections $K^{\natural}{}_{\mathrm{cen}}$ and $K^{\natural}{}_{\mathrm{rel}}$ preserve the metrics g_{cen} and g_{rel}.

The splitting of the multi-spacetime yields a splitting of the multi-electromagnetic field. It can be readily proved that $\mathcal{F}_{\mathrm{mul}}$ splits into the three components

$$\mathcal{F}_{\mathrm{mul}} = \mathcal{F}_{\mathrm{cen}} + \mathcal{F}_{\mathrm{rel}} + \mathcal{F}_{\mathrm{cenrel}}, \tag{3}$$

where

$$\mathcal{F}_{\mathrm{cen}} : \boldsymbol{E}_{\mathrm{mul}} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E}_{\mathrm{cen}},$$
$$\mathcal{F}_{\mathrm{rel}} : \boldsymbol{E}_{\mathrm{mul}} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{S}_{\mathrm{rel}},$$
$$\mathcal{F}_{\mathrm{cenrel}} : \boldsymbol{E}_{\mathrm{mul}} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes (T^* \boldsymbol{E}_{\mathrm{cen}} \wedge T^* \boldsymbol{S}_{\mathrm{rel}}).$$

Of course, the potential $\mathcal{A}_{\mathrm{mul}}$ for $\mathcal{F}_{\mathrm{mul}}$ splits in an analogous way

$$\mathcal{A}_{\mathrm{mul}} = \mathcal{A}_{\mathrm{cen}} + \mathcal{A}_{\mathrm{rel}}, \quad \text{where} \quad \mathcal{A}_{\mathrm{cen}} : \boldsymbol{E}_{\mathrm{mul}} \to T^* \boldsymbol{E}_{\mathrm{cen}}, \quad \mathcal{A}_{\mathrm{rel}} : \boldsymbol{E}_{\mathrm{mul}} \to T^* \boldsymbol{E}_{\mathrm{rel}},$$

with $\mathcal{A}_{\mathrm{cen}}(e_{\mathrm{mul}}; v_{\mathrm{mul}}) = \sum_i \frac{q_i}{m} A_i(e_i; v_{\mathrm{cen}\,i})$ and $\mathcal{A}_{\mathrm{rel}}(e_{\mathrm{mul}}; v_{\mathrm{mul}}) = \sum_i \frac{q_i}{m} A_i(e_i; v_{\mathrm{rel}\,i})$.

We stress that, in general, each of the three components of the multi-electromagnetic field depends on the whole multi-spacetime and not just on the corresponding components. As a consequence, in general, the equation of motion of the multi-particle splits into a system of equations for the motion of the centre of mass and for the relative multi-motion. But those equations are generally coupled, with the exception of particular cases, like when the pattern electromagnetic field F is constant and the charges are proportional to the masses (i.e., $q_i = k m_i$; in this case the mixed term $\mathcal{F}_{\mathrm{cenrel}}$ vanishes).

2.3. Rigid body mechanics

2.3.1. Configuration space. To carry on our analysis, we need a 'generalized' definition of affine space. Namely, we define a *generalized affine space* to be a triple $(A, G, \cdot)$, where A is a set, G a group and $\cdot$ a transitive and free left action of G on A. Note that, for every $a \in A$, the 'left translation' $L(a) : G \to A : g \mapsto ga$ is a bijection. The generalized affine space A is naturally parallelizable as $TA = A \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G.

We consider a set $\{l_{ij} \in \mathbb{L}^2 | i, j = 1, \ldots, n, i \neq j, l_{ij} = l_{ji}, l_{ik} \leqslant l_{ij} + l_{jk}\}$ and define the subsets

$$i_{\text{rig}} : \boldsymbol{E}_{\text{rig}} := \{e_{\text{mul}} \in \boldsymbol{E}_{\text{mul}} | \|e_i - e_j\| = l_{ij}, 1 \leqslant i < j \leqslant n\} \hookrightarrow \boldsymbol{E}_{\text{mul}},$$
$$i_{\text{rot}} : \boldsymbol{S}_{\text{rot}} := \{v_{\text{rel}} \in \boldsymbol{S}_{\text{rel}} | \|v_i - v_j\| = l_{ij}, 1 \leqslant i < j \leqslant n\} \hookrightarrow \boldsymbol{S}_{\text{rel}}.$$

We set $\boldsymbol{E}_{\text{rot}} := \boldsymbol{T} \times \boldsymbol{S}_{\text{rot}}$. We stress that the rigid constraint does not affect the centre of mass.

Then, the splitting (2) restricts to a splitting

$$\boldsymbol{E}_{\text{rig}} = \boldsymbol{E}_{\text{cen}} \underset{\boldsymbol{T}}{\times} \boldsymbol{E}_{\text{rot}} = \boldsymbol{E}_{\text{cen}} \times \boldsymbol{S}_{\text{rot}}. \tag{4}$$

Thus, we obtain a curved fibred manifold $t_{\text{rig}} : \boldsymbol{E}_{\text{rig}} \to \boldsymbol{T}$. The first jet space of $\boldsymbol{E}_{\text{rig}}$ splits as $J_1\boldsymbol{E}_{\text{rig}} \simeq (\boldsymbol{E}_{\text{cen}} \times \boldsymbol{U}_{\text{cen}}) \times (\mathbb{T}^* \otimes T\boldsymbol{S}_{\text{rot}})$. Each rigid observer $o : \boldsymbol{E} \to J_1\boldsymbol{E}$, induces an observer $o_{\text{rig}} : \boldsymbol{E}_{\text{rig}} \to J_1\boldsymbol{E}_{\text{rig}}$. In particular, each inertial observer $o \in \boldsymbol{U}$ induces an observer $o_{\text{rig}} \in \boldsymbol{U}_{\text{cen}}$, which is still called *inertial*.

The inclusion i_{rig} yields the scaled space-like Riemannian metric

$$g_{\text{rig}} := i^*_{\text{rig}} g_{\text{mul}} : V\boldsymbol{E}_{\text{rig}} \underset{\boldsymbol{E}_{\text{rig}}}{\times} V\boldsymbol{E}_{\text{rig}} \to \mathbb{L}^2 \otimes \mathbb{R}.$$

In order to further analyse the geometry of $\boldsymbol{E}_{\text{rig}}$, it suffices to study $\boldsymbol{S}_{\text{rot}}$.

The geometry of $\boldsymbol{S}_{\text{rot}}$ depends on the initial mutual positions of particles and is time independent. In particular, particles can either lie on a straight line, or lie on a plane, or 'span' the whole space. This can be formalized as follows.

For each $r_{\text{rot}} \in \boldsymbol{S}_{\text{rot}}$, let us consider the vector space

$$\langle r_{\text{rot}} \rangle := \text{span}\,\{(r_i - r_j) | 1 \leqslant i < j \leqslant n\} \subset \boldsymbol{S}.$$

We can prove that the dimension of this space depends only on $\boldsymbol{S}_{\text{rot}}$ and not on the choice of $r_{\text{rot}} \in \boldsymbol{S}_{\text{rig}}$. We call this invariant number c_{rot} the *characteristic* of $\boldsymbol{S}_{\text{rot}}$. We can have $c_{\text{rot}} = 1, 2, 3$. We say that $\boldsymbol{S}_{\text{rot}}$ is *strongly non-degenerate* if $c_{\text{rot}} = 3$, *weakly non-degenerate* if $c_{\text{rot}} = 2$, *degenerate* if $c_{\text{rot}} = 1$.

We observe that the natural componentwise action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\text{rel}}$ restricts to a transitive action on $\boldsymbol{S}_{\text{rot}}$. For each $v_{\text{rot}} \in \boldsymbol{S}_{\text{rot}}$, let us call $H[r_{\text{rot}}] \subset O(\boldsymbol{S}, g)$ the corresponding isotropy subgroup. We can see that

- in the *strongly non-degenerate case* the isotropy subgroup $H[r_{\text{rot}}]$ is trivial;
- in the *weakly non-degenerate case* the isotropy subgroup $H[r_{\text{rot}}]$ is the discrete subgroup of reflections with respect to $\langle v_{\text{rot}} \rangle$;
- in the *degenerate case* the isotropy subgroup $H[r_{\text{rot}}]$ is the one-dimensional subgroup of rotations about the line $\langle r_{\text{rot}} \rangle$; we stress that this subgroup is not normal.

Hence, we can prove that

- $\boldsymbol{S}_{\text{rot}}$ is *strongly non-degenerate* if and only if the action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\text{rot}}$ is free;
- $\boldsymbol{S}_{\text{rot}}$ is *weakly non-degenerate* if and only if the action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\text{rot}}$ is not free, but the action of $SO(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\text{rot}}$ is free;

– $\boldsymbol{S}_{\mathrm{rot}}$ is *degenerate* if and only if the action of $SO(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\mathrm{rot}}$ is not free.
  Of course, if $n = 2$, then $\boldsymbol{S}_{\mathrm{rot}}$ is degenerate; if $n = 3$, then $\boldsymbol{S}_{\mathrm{rot}}$ can be degenerate or weakly non-degenerate.

Furthermore, we can prove that

– if $\boldsymbol{S}_{\mathrm{rot}}$ is *strongly non-degenerate*, then $\boldsymbol{S}_{\mathrm{rot}}$ is an affine space associated with the group $O(\boldsymbol{S}, g)$;
– if $\boldsymbol{S}_{\mathrm{rot}}$ is *weakly non-degenerate*, then $\boldsymbol{S}_{\mathrm{rot}}$ is an affine space associated with the group $SO(\boldsymbol{S}, g)$;
– if $\boldsymbol{S}_{\mathrm{rot}}$ is *degenerate*, then $\boldsymbol{S}_{\mathrm{rot}}$ is a homogeneous manifold with two possible distinguished diffeomorphisms (depending on a chosen orientation on the straight line of the rigid body) with the unit sphere $S^2(\mathbb{L}^* \otimes \boldsymbol{S}, g)$.

So, the choice of a configuration $v_{\mathrm{rot}} \in \boldsymbol{S}_{\mathrm{rot}}$ and of a scaled orthonormal basis in $\boldsymbol{S}$, respectively, yields the following diffeomorphisms (via the action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\mathrm{rot}}$):

$$\boldsymbol{S}_{\mathrm{rot}} \simeq O(3), \qquad \text{in the strongly non-degenerate case;} \tag{5}$$

$$\boldsymbol{S}_{\mathrm{rot}} \simeq SO(3), \qquad \text{in the weakly non-degenerate case;} \tag{6}$$

$$\boldsymbol{S}_{\mathrm{rot}} \simeq S^2, \qquad \text{in the degenerate case,} \tag{7}$$

where $S^2 \subset \mathbb{L}^* \otimes \boldsymbol{S}$ is the unit sphere with respect to the metric g.

From now on, for the sake of simplicity and for physical reasons of continuity, in the non-degenerate case, we shall refer only to one of the two connected components of $\boldsymbol{S}_{\mathrm{rot}}$. Accordingly, we shall just refer to the non-degenerate case (without specification of strongly or weakly non-degenerate) or to the degenerate case.

Proposition 2.1. *In the non-degenerate case, by considering the isomorphism* $\boldsymbol{S}_{\mathrm{rot}} \simeq SO(3)$ *and the well-known two-fold universal covering* $S^3 \simeq SU(2) \to SO(3)$, *we obtain the universal covering* $S^3 \to \boldsymbol{S}_{\mathrm{rot}}$, *which is a principal bundle associated with the group* $\mathbb{Z}_2$ *[33, vol 1].*

This is in agreement with the fact that the homotopy group of $\boldsymbol{E}_{\mathrm{rot}}$ is [15, vol 2] $\pi_1(\boldsymbol{E}_{\mathrm{rot}}) = \mathbb{Z}_2$.

2.3.2. Tangent space of rotational space.

Non-degenerate case. The generalized affine structure of $\boldsymbol{S}_{\mathrm{rot}}$, with respect to the group $O(\boldsymbol{S}, g)$, yields the natural parallelization $T\boldsymbol{S}_{\mathrm{rot}} = \boldsymbol{S}_{\mathrm{rot}} \times \mathfrak{so}(\boldsymbol{S}, g)$. We can regard this isomorphism in another interesting way, which expresses in a geometric language the classical formula of the velocity of a rigid body. For this purpose, let us consider the three-dimensional scaled vector space $\boldsymbol{V}_{\mathrm{ang}} := \mathbb{L}^* \otimes \boldsymbol{S}$. Then, the metric g and the chosen orientation of $\boldsymbol{S}$ determine the linear isomorphisms $g^\flat : \mathfrak{so}(\boldsymbol{S}, g) \to \mathbb{L}^2 \otimes \Lambda^2 \boldsymbol{S}^*$ and $* : \mathbb{L}^2 \otimes \Lambda^2 \boldsymbol{S}^* \to \boldsymbol{V}_{\mathrm{ang}}$, hence the linear isomorphism $\mathfrak{so}(\boldsymbol{S}, g) \simeq \boldsymbol{V}_{\mathrm{ang}}$. Therefore, we can read the above parallelization also as

$$\tau_{\mathrm{ang}} : T\boldsymbol{S}_{\mathrm{rot}} \simeq \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}}. \tag{8}$$

The inverse $\tau_{\mathrm{ang}}^{-1} : \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}} \to T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{rel}}$ of the above isomorphism is expressed by the formula

$$(r_1, \ldots, r_n; \omega) \mapsto (r_1, \ldots, r_n; \omega \times r_1, \ldots, \omega \times r_1),$$

where $\times$ is the cross product of $\boldsymbol{S}$ defined by $u \times v := g^\sharp(\mathrm{i}(u \wedge v)\eta)$, where η is the metric volume form of $\boldsymbol{S}$. The above formula is just a geometric formulation of the well-known formula expressing the relative velocity of the particles of a rigid body through the angular velocity. In other words, for each $(r_1, \ldots, r_n, v_1, \ldots, v_n) \in T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{rel}}$, there is a unique $\omega \in \boldsymbol{V}_{\mathrm{ang}}$ such that $v_i = \omega \times r_i$, for $1 \leqslant i \leqslant n$. Note that the cross product $\times$ of $\boldsymbol{S}$ is equivariant with respect to the left action of $SO(\boldsymbol{S}, g)$, hence, the isomorphism τ_{ang} turns out to be equivariant with respect to this group.

The *angular velocity* of a rigid motion $s : \boldsymbol{T} \to \boldsymbol{E}_{\mathrm{rig}}$ is defined to be the map

$$\omega := \tau_{\mathrm{ang}} \circ T\pi_{\mathrm{rot}} \circ ds : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{V}_{\mathrm{ang}},$$

where $\pi_{\mathrm{rot}} : \boldsymbol{E}_{\mathrm{rig}} \to \boldsymbol{S}_{\mathrm{rot}}$ is the natural projection map according to section 2.3.1.

We stress that the above geometric constructions use implicitly the pattern affine structure. Hence, the angular velocity is independent of the choice of inertial observers. But, the observed angular velocity would depend on the choice of non-inertial observers.

Degenerate case. According to a well-known result on homogeneous spaces, we have $T\boldsymbol{S}_{\mathrm{rot}} = \boldsymbol{S}_{\mathrm{rot}} \times \mathfrak{so}(\boldsymbol{S}, g)/h[\boldsymbol{S}_{\mathrm{rot}}]$, where $h[\boldsymbol{S}_{\mathrm{rot}}] \subset \boldsymbol{S}_{\mathrm{rot}} \times \mathfrak{so}(\boldsymbol{S}, g)$ is the vector subbundle over $\boldsymbol{S}_{\mathrm{rot}}$ consisting of the isotropy Lie algebras of $\boldsymbol{S}_{\mathrm{rot}}$.

Now, we define the quotient vector bundle $(\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}})/\sim$, over $\boldsymbol{S}_{\mathrm{rot}}$ induced, for each $r_{\mathrm{rot}} \in \boldsymbol{S}_{\mathrm{rot}}$, by the vector subspace $\langle r_{\mathrm{rot}} \rangle \subset \boldsymbol{V}_{\mathrm{ang}}$ generated by r_{rot}. Then, by proceeding as in the non-degenerate case and taking the quotient with respect to the isotropy subbundle, we obtain the linear fibred isomorphism

$$[\tau_{\mathrm{ang}}] : T\boldsymbol{S}_{\mathrm{rot}} \simeq (\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}})/\sim,$$

and the inverse $[\tau_{\mathrm{ang}}]^{-1}$ is expressed by the same formula as the non-degenerate case. Note that the cross products $\omega \times r_i$ turns out to be independent of the choice of representative for the class $[\omega]$.

Clearly, each choice of the orientation of the rigid body yields a distinguished fibred isomorphism $T\boldsymbol{S}_{\mathrm{rot}} \simeq TS^2(\mathbb{L}^* \otimes \boldsymbol{S}, g)$ with the unit sphere.

Induced metrics. The multi-metric of $\boldsymbol{S}_{\mathrm{mul}}$ induces a metric on $\boldsymbol{S}_{\mathrm{rot}}$, which can also be regarded in another useful way through the isomorphism τ_{ang}.

Moreover, the standard pattern metric of $\boldsymbol{V}_{\mathrm{ang}}$ induces a further metric on $\boldsymbol{S}_{\mathrm{rot}}$, which will be interpreted as the inertia tensor.

The inclusion i_{rot} yields the scaled Riemannian metric $g_{\mathrm{rot}} := i^*_{\mathrm{rot}} g_{\mathrm{rel}}$. We can regard this metric in another interesting way, which follows from the parallelization τ_{ang}:

$$\sigma := \tau^{-1*}_{\mathrm{ang}} g_{\mathrm{rot}} : \boldsymbol{S}_{\mathrm{rot}} \times (\boldsymbol{V}_{\mathrm{ang}} \times \boldsymbol{V}_{\mathrm{ang}}) \to \mathbb{L}^2 \otimes \mathbb{R}$$
$$\sigma := \tau^{-1*}_{\mathrm{ang}} g_{\mathrm{rot}} : (\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}})/\sim) \underset{\boldsymbol{S}_{\mathrm{rot}}}{\times} (\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}})/\sim) \to \mathbb{L}^2 \otimes \mathbb{R},$$

respectively, in the non-degenerate and in the degenerate cases, with expressions

$$\sigma(r_1, \ldots, r_n; \omega, \omega') = \sum_i \mu_i (g(r_i, r_i) g(\omega, \omega') - g(r_i, \omega) g(r_i, \omega')) \tag{9}$$

$$\sigma(r_1, \ldots, r_n; [\omega], [\omega']) = \sum_i \mu_i (g(r_i, r_i) g(\omega, \omega') - g(r_i, \omega) g(r_i, \omega')). \tag{10}$$

In the degenerate case, the right-hand side of (10) is equal to $g(\omega, \omega') \sum_i \mu_i g(r_i, r_i)$, where ω and ω' are the representatives of $[\omega]$ and $[\omega']$ orthogonal to the r_i's.

Now, we observe that the pattern metric g can be regarded as a Euclidean metric on $\boldsymbol{V}_{\mathrm{ang}}$. This yields a fibred metric over $\boldsymbol{S}_{\mathrm{rot}}$, which will be denoted by the same symbol

$$g : \boldsymbol{S}_{\mathrm{rot}} \times (\boldsymbol{V}_{\mathrm{ang}} \times \boldsymbol{V}_{\mathrm{ang}}) \to \mathbb{R}$$
$$g : (\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}})/\sim) \underset{\boldsymbol{S}_{\mathrm{rot}}}{\times} (\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}})/\sim) \to \mathbb{R},$$

respectively, in the non-degenerate and in the degenerate cases, according to the equalities

$$g(r_1, \dots, r_n; \omega, \omega') = g(\omega, \omega')$$
$$g(r_1, \dots, r_n; [\omega], [\omega']) = g(\omega_\perp, \omega'_\perp),$$

where $\omega_\perp$ and $\omega'_\perp$ are the components of ω and ω' orthogonal to r_i. Then, we obtain the further unscaled Riemannian metric $\sigma_{\mathrm{rot}} := \tau^*_{\mathrm{ang}} g$ of $\boldsymbol{S}_{\mathrm{rot}}$.

All metrics of $\boldsymbol{S}_{\mathrm{rot}}$ considered above are invariant with respect to the left action of $O(\boldsymbol{S}, g)$. As a straightforward consequence, we have that the diffeomorphisms (5), (6) turn out to be isometries with respect to the metrics $\sigma_{\mathrm{rot}}, -\frac{1}{2} k_3$, where k_3 is the Killing form, and the diffeomorphism (7) turns out to be an isometry with respect to the metrics σ_{rot} and g_{S^2}, where g_{S^2} is the standard Riemannian metric on the sphere S^2.

Inertia tensor. The fibred metric g of $\boldsymbol{S}_{\mathrm{rot}}$ allows us to regard the fibred metric σ of $\boldsymbol{S}_{\mathrm{rot}}$ as a scaled symmetric fibred automorphism

$$\hat{\sigma} : \boldsymbol{S}_{\mathrm{rot}} \to \mathbb{L}^2 \otimes (\boldsymbol{V}^*_{\mathrm{ang}} \times \boldsymbol{V}_{\mathrm{ang}}).$$

The scaled metric $m\sigma$, or the scaled automorphism $m\hat{\sigma}$, are called the *inertia tensor*. The scaled eigenvalues of the inertia tensor are called *principal inertia momenta* and are denoted by $I_i \in \mathrm{map}(\boldsymbol{S}_{\mathrm{rot}}, \mathbb{L}^2 \otimes \mathbb{M} \otimes \mathbb{R})$. Indeed, the principal inertia momenta turn out to be constant with respect to $\boldsymbol{S}_{\mathrm{rot}}$.

In the non-degenerate case, we have three principal inertia momenta. Then, three cases can occur:

$$I := I_1 = I_2 = I_3, \qquad \text{spherical case,}$$
$$I := I_1 = I_2 \neq I_3, \qquad \text{symmetric case,}$$
$$I_1 \neq I_2 \neq I_3 \neq I_1, \qquad \text{asymmetric case.}$$

In the degenerate case, we have $I := I_1 = I_2 = \sum_i m_i g(r_i, r_i)$. Hence, both in the spherical non-degenerate case and in the degenerate case we have $g_{\mathrm{rot}} = (I/m)\sigma$.

Thus, we have studied the diagonalization of σ with respect to g. In an analogous way, we can diagonalize g_{rot} with respect to σ_{rot}. Indeed, in this way we obtain the same eigenvalues and the same classification, because the two diagonalizations are related by the isomorphism τ_{ang}.

The principal inertia momenta are related to the scalar curvature of the rotational space in the following way.

Proposition 2.2. *The scalar curvature of $\boldsymbol{S}_{\mathrm{rot}}$, with respect to the metric G_{rot}, is [53]*

$$\rho_{\mathrm{rot}} = \frac{3\hbar}{2I}, \qquad \textit{spherical non deg. case}$$
$$\rho_{\mathrm{rot}} = \frac{2\hbar}{I_3} - \frac{\hbar I}{2I_3^2}, \qquad \textit{symmetric non deg. case}$$
$$\rho_{\mathrm{rot}} = \frac{\hbar}{I_1} + \frac{\hbar}{I_2} + \frac{\hbar}{I_3} - \frac{\hbar\left(I_1^2 + I_2^2 + I_3^2\right)}{2I_1 I_2 I_3}, \qquad \textit{asymmetric non deg. case}$$
$$\rho_{\mathrm{rot}} = \frac{2\hbar}{I}, \qquad \textit{degenerate case}$$

Moreover, since the splitting $\boldsymbol{E}_{\text{rig}} = \boldsymbol{E}_{\text{cen}} \times \boldsymbol{S}_{\text{rot}}$ *is orthogonal, the vanishing of the scalar curvature* ρ_{cen} *yields* $\rho_{\text{rig}} = \rho_{\text{rot}}$.

2.3.3. Fields and dynamics. The multi-connection of the multi-spacetime induces naturally a connection on the rigid configuration space, which splits naturally into the centre of mass and relative components.

Moreover, the pull-back of the electromagnetic field on the rigid configuration space splits into three components: the centre of mass component, the rotational component and the mixed component. Indeed, the pullback of the multi-electromagnetic field on the rigid spacetime provides the suitable electromagnetic object for the correct expression of the classical law of motion (in the context of our formulation of classical mechanics of a rigid body interpreted as a classical particle moving in a higher dimensional spacetime).

Induced fields. We can easily state the following generalization of a well-known theorem due to Gauss [33].

Lemma 2.3. *Let us consider a fibred manifold* $p : \boldsymbol{F} \to \boldsymbol{B}$ *equipped with a vertical Riemannian metric* $g_{\boldsymbol{F}}$ *and a linear connection* $K_{\boldsymbol{F}}$ *of* $\boldsymbol{F}$, *which restricts to the fibres of* $\boldsymbol{F} \to \boldsymbol{B}$ *and preserves the metric* $g_{\boldsymbol{F}}$.

Moreover, let us consider a fibred submanifold $\boldsymbol{G} \subset \boldsymbol{F}$ *over* $\boldsymbol{B}$ *and the orthogonal projection* $\pi_{\boldsymbol{G}} : T\boldsymbol{F}_{|\boldsymbol{G}} \to T\boldsymbol{G}$ *induced by* $g_{\boldsymbol{F}}$. *Then, there exists a unique linear connection* $K_{\boldsymbol{G}}$ *of* $\boldsymbol{G}$, *which restricts to the fibres of* $\boldsymbol{G} \to \boldsymbol{B}$ *and such that, for every pair of vector fields* X, Y *of* $\boldsymbol{G}$, *we have* $\pi_{\boldsymbol{G}}(\nabla[K_{\boldsymbol{F}}]_X Y) = \nabla[K_{\boldsymbol{G}}]_X Y$. *Moreover, this connection* $K_{\boldsymbol{G}}$ *preserves* $g_{\boldsymbol{G}}$.

According to the above Lemma, the connection $K^{\natural}{}_{\text{mul}}$ of $\boldsymbol{E}_{\text{mul}}$ yields a linear connection $K^{\natural}{}_{\text{rig}}$ of $\boldsymbol{E}_{\text{rig}}$, which preserves the time fibring and the metric g_{rig}. Moreover, according to a standard result due to Gauss, the connection $K^{\natural}{}_{\text{rel}}$ of $\boldsymbol{S}_{\text{rel}}$ induces a connection $\varkappa^{\natural}{}_{\text{rot}}$ on $\boldsymbol{S}_{\text{rot}}$, which coincides with the Riemannian connection induced by g_{rot}.

Proposition 2.4. *By considering the splitting* $\boldsymbol{E}_{\text{rig}} = \boldsymbol{E}_{\text{cen}} \times \boldsymbol{S}_{\text{rot}}$, *the connection* $K^{\natural}{}_{\text{rig}}$ *splits into the product of the connections* $K^{\natural}{}_{\text{cen}}$ *and* $\varkappa^{\natural}{}_{\text{rot}}$.

Proof. We have the splitting $K^{\natural}{}_{\text{mul}} = K^{\natural}{}_{\text{cen}} \times K^{\natural}{}_{\text{rel}}$. Moreover, the splitting $\boldsymbol{E}_{\text{mul}} = \boldsymbol{E}_{\text{cen}} \times \boldsymbol{S}_{\text{rel}}$ is orthogonal with respect to the metric g_{mul}, hence the projection $\pi_{\boldsymbol{E}_{\text{rig}}}$ splits into the projections $\boldsymbol{E}_{\text{mul}} \to \boldsymbol{E}_{\text{cen}}$ and $\boldsymbol{E}_{\text{mul}} \to \boldsymbol{S}_{\text{rel}}$. Hence, $K^{\natural}{}_{\text{rig}}$ splits into the product of the connections $K^{\natural}{}_{\text{cen}}$ and $\varkappa_{\text{rot}}$. □

Now, let us analyse the splitting of the electromagnetic field. In the non-degenerate case the inclusion $i_{\text{rig}} : \boldsymbol{E}_{\text{rig}} = \boldsymbol{E}_{\text{cen}} \times \boldsymbol{S}_{\text{rot}} \hookrightarrow \boldsymbol{E}_{\text{mul}}$ yields the scaled 2-form $\mathcal{F}_{\text{rig}} := i^*_{\text{rig}} \mathcal{F}_{\text{mul}}$, which splits into the three components

$$\mathcal{F}_{\text{rig}} = \mathcal{F}_{\text{cen}} + \mathcal{F}_{\text{rot}} + \mathcal{F}_{\text{cenrot}}, \tag{11}$$

analogously to (3). Accordingly, the potential $\mathcal{A}_{\text{rig}} := i^*_{\text{rig}} \mathcal{A}_{\text{mul}}$ for $\mathcal{F}_{\text{rig}}$ splits as $\mathcal{A}_{\text{rig}} = \mathcal{A}_{\text{cen}} + \mathcal{A}_{\text{rot}}$.

The degenerate case can be studied in a similar way to the non-degenerate one.

Dynamics. We assume the fibred manifold $t_{\text{rig}} : \boldsymbol{E}_{\text{rig}} \to \boldsymbol{T}$ as *rigid-body spacetime*. Moreover, we assume the metric $g_{\text{rig}} := i^*_{\text{rig}} g_{\text{mul}}$ as the *space-like metric*, the metric $G_{\text{rig}} = \frac{m}{\hbar}\, g_{\text{rig}}$ as the *rescaled space-like metric*, the connection $K^{\natural}{}_{\text{rig}} = K^{\natural}{}_{\text{cen}} \times \varkappa^{\natural}{}_{\text{rig}}$ as the *gravitational connection*, the 2-form $\mathcal{F}_{\text{rig}} := i^*_{\text{rig}} \mathcal{F}_{\text{mul}}$ as the *rescaled electromagnetic field*, and the 2-form $\frac{m}{\hbar}\, \mathcal{F}_{\text{rig}}$ as the *unscaled electromagnetic field*.

The joined cosymplectic 2-form Ω_{rig} (induced by the above gravitational connection, the unscaled electromagnetic field and the rescaled metric) coincides with the pullback $\Omega_{\text{rig}} = i^*\Omega_{\text{mul}}$. Hence, Ω_{rig} turns out to be a globally exact cosymplectic 2-form.

The velocity space of $\boldsymbol{E}_{\text{rig}}$ splits as $J_1\boldsymbol{E}_{\text{rig}} \simeq (\boldsymbol{E}_{\text{cen}} \times \boldsymbol{U}_{\text{cen}}) \times (\mathbb{T}^* \otimes T\boldsymbol{S}_{\text{rot}})$.

An inertial observer o yields the further splitting $\boldsymbol{E}_{\text{cen}} = \boldsymbol{T} \times \boldsymbol{P}[o]_{\text{cen}}$.

Given an inertial observer o, we shall refer to a spacetime chart (x^0, x^i, x^α) adapted to the observer and to the centre-of-mass splitting. Here, indices i, j will label coordinates of $\boldsymbol{P}[o]_{\text{cen}}$ and α, β will label coordinates of $\boldsymbol{S}_{\text{rot}}$ (e.g., Euler angles).

Now, we discuss the momentum and Hamiltonian functions and their splitting into the translational and rotational components. Let us choose a horizontal potential $A^{\uparrow}{}_{\text{rig}}$ for Ω_{rig} and an inertial observer o. They yield the rigid momentum and Hamiltonian

$$\mathcal{P}_{\text{rig}} := \nu[o] \lrcorner A^{\uparrow}{}_{\text{rig}} : J_1\boldsymbol{E}_{\text{rig}} \to T^*\boldsymbol{E}_{\text{rig}}, \qquad \mathcal{H}_{\text{rig}} := -o \lrcorner A^{\uparrow}{}_{\text{rig}} : J_1\boldsymbol{E}_{\text{rig}} \to T^*\boldsymbol{E}_{\text{rig}},$$

which split as

$$\mathcal{P}_{\text{rig}} = \mathcal{P}_{\text{cen}} + \mathcal{P}_{\text{rot}}, \qquad \mathcal{H}_{\text{rig}} = \mathcal{H}_{\text{cen}} + \mathcal{H}_{\text{rot}},$$

where

$$\begin{aligned} \mathcal{P}_{\text{cen}} &: J_1\boldsymbol{E}_{\text{rig}} \to T^*\boldsymbol{E}_{\text{cen}}, & \mathcal{P}_{\text{rot}} &: J_1\boldsymbol{E}_{\text{rig}} \to T^*\boldsymbol{E}_{\text{rot}}, \\ \mathcal{H}_{\text{cen}} &: J_1\boldsymbol{E}_{\text{rig}} \to T^*\boldsymbol{E}_{\text{cen}}, & \mathcal{H}_{\text{rot}} &: J_1\boldsymbol{E}_{\text{rig}} \to T^*\boldsymbol{E}_{\text{rot}}. \end{aligned}$$

We have the coordinate expressions

$$\begin{aligned} \mathcal{P}_{\text{cen}\, j} &= G_{\text{cen}}{}^{0}_{ij}\, x^j_0 + A_{\text{cen}\, i}, & \mathcal{P}_{\text{rot}\, \alpha} &= G_{\text{rot}}{}^{0}_{\alpha\beta}\, x^\beta_0 + A_{\text{rot}\, \alpha}, \\ \mathcal{H}_{\text{cen}\, 0} &= \tfrac{1}{2}\, G_{\text{cen}}{}^{0}_{ij}\, x^i_0 x^j_0 - A_{\text{cen}\, 0}, & \mathcal{H}_{\text{rot}\, 0} &= \tfrac{1}{2}\, G_{\text{rot}}{}^{0}_{\alpha\beta}\, x^\alpha_0 x^\beta_0. \end{aligned}$$

Clearly, $\mathcal{P}_{\text{cen}}$ and $\mathcal{P}_{\text{rot}}$ can be identified with the angular momentum of the centre of mass and the angular momentum with respect to the centre of mass, respectively.

In the general case the above quantities are coupled and not conserved. But in the particular case when $F = 0$, they are conserved and we obtain decoupled expressions.

3. Rigid body quantum mechanics

We approach the quantization of the rigid body according to the scheme of 'covariant quantum mechanics', by analogy with the case of a one body.

We define quantum structures, analyse their existence and classify them. Then, we evaluate the quantum operators and compute the spectra of the energy operator in some cases.

3.1. Quantum structures

According to proposition 1.1, the existence condition of quantum structures is fulfilled due to the exactness of $\Omega_{\text{rig}} : [\Omega] = 0 \in i^2\big(H^2(\boldsymbol{E}, \mathbb{Z})\big) \subset H^2(\boldsymbol{E}, \mathbb{R})$. So, we have just to compute all possible inequivalent quantum structures.

Non-degenerate case. Let us start with the non-degenerate case.

Proposition 3.1. *We have just two equivalence classes of complex line bundles over* $\boldsymbol{E}_{\mathrm{rig}}$*. Clearly, one of these classes is the trivial one. Indeed, both of them admit quantum connections.*

Proof. The second cohomology groups of $\boldsymbol{E}_{\mathrm{rig}}$ are [6]:

$$H^2(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{Z}) \simeq H^2(\boldsymbol{S}_{\mathrm{rig}}, \mathbb{Z}) \simeq H^2(SO(3), \mathbb{Z}) \simeq \mathbb{Z}_2,$$
$$H^2(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{R}) \simeq H^2(\boldsymbol{S}_{\mathrm{rig}}, \mathbb{R}) \simeq H^2(SO(3), \mathbb{R}) \simeq \{0\}.$$

Then, according to proposition 1.1, the equivalence classes of complex line bundles are in bijection with $H^2(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{Z}) = \mathbb{Z}_2$ and the equivalence classes of quantum bundles are in bijection with $(i^2)^{-1}([\Omega]) = (i^2)^{-1}(0) = \mathbb{Z}_2$. □

We can produce two concrete representatives for the above equivalence classes of vector bundles in the following way.

Lemma 3.2. *The two inequivalent representations of* $\mathbb{Z}_2$ *on* $\mathbb{C}$ *yield the trivial Hermitian line bundle* $\boldsymbol{Q}^+_{\mathrm{rot}}$ *and the non-trivial Hermitian line bundle* $\boldsymbol{Q}^-_{\mathrm{rot}}$*, equipped with flat Hermitian connections* χ^+_{rot} *and* χ^-_{rot}*, respectively.*

These bundles admit an atlas with constant transition maps and the above flat connections have vanishing symbols with respect to this atlas.

Proof. Let us consider the two inequivalent representations of $\mathbb{Z}_2$ on $\mathbb{C}$

$$\rho^+(1) = 1, \quad \rho^+(-1) = 1 \qquad \text{and} \qquad \rho^-(1) = 1, \quad \rho^-(-1) = -1.$$

Then, the quotient of the trivial Hermitian line bundle $\tilde{\boldsymbol{Q}}_{\mathrm{rot}} = S^3 \times \mathbb{C} \to S^3$ with respect to the above actions of $\mathbb{Z}_2$ yields, respectively, the associated trivial and non-trivial Hermitian line bundles over $\boldsymbol{S}_{\mathrm{rot}}$:

$$\boldsymbol{Q}^+_{\mathrm{rot}} = S^3 \underset{\rho^+}{\times} \mathbb{C} \qquad \text{and} \qquad \boldsymbol{Q}^-_{\mathrm{rot}} = S^3 \underset{\rho^-}{\times} \mathbb{C}.$$

Moreover, the natural flat principal connection of the principal bundle $S^3 \to SO(3)$ yields two flat Hermitian connections χ^+_{rot} and χ^-_{rot} on $\boldsymbol{Q}^+_{\mathrm{rot}}$ and $\boldsymbol{Q}^-_{\mathrm{rot}}$, respectively. □

Proposition 3.3. *The pullback with respect to the projection* $\boldsymbol{E}_{\mathrm{rig}} \to \boldsymbol{S}_{\mathrm{rot}}$ *yields a trivial and a non-trivial Hermitian line bundle*

$$\boldsymbol{Q}^+_{\mathrm{rig}} \to \boldsymbol{E}_{\mathrm{rig}} \qquad \text{and} \qquad \boldsymbol{Q}^-_{\mathrm{rig}} \to \boldsymbol{E}_{\mathrm{rig}},$$

which are equipped with the pullback flat Hermitian connections χ^+_{rig} *and* χ^-_{rig}*, respectively.*

Theorem 3.4. *Let* $\boldsymbol{E}_{\mathrm{rig}}$ *be non-degenerate. Then, the only inequivalent quantum structures are of the type* $\big(\boldsymbol{Q}^{\uparrow +}_{\mathrm{rig}}, \text{Ч}^{\uparrow +}_{\mathrm{rig}}\big)$ *and* $(\boldsymbol{Q}^{\uparrow -}_{\mathrm{rig}}, \text{Ч}^{\uparrow -}_{\mathrm{rig}})$*, with*

$$\text{Ч}^{\uparrow +}_{\mathrm{rig}} = \chi^{\uparrow +}_{\mathrm{rig}} + \mathrm{i}\, A^{\uparrow +}_{\mathrm{rig}} \otimes \mathbb{I}^{\uparrow} \qquad \text{and} \qquad \text{Ч}^{\uparrow -}_{\mathrm{rig}} = \chi^{\uparrow -}_{\mathrm{rig}} + \mathrm{i}\, A^{\uparrow -}_{\mathrm{rig}} \otimes \mathbb{I}^{\uparrow},$$

where $\chi^{\uparrow +}_{\mathrm{rig}}, \chi^{\uparrow -}_{\mathrm{rig}}$ *are the pullbacks of* $\chi^+_{\mathrm{rig}}, \chi^-_{\mathrm{rig}}$*, and* $A^{\uparrow +}_{\mathrm{rig}}, A^{\uparrow -}_{\mathrm{rig}}$ *are two global horizontal potentials for* Ω*.*

Proof. According to proposition 1.1, inequivalent quantum structures are in bijection with the set

$$(i^2)^{-1}([\Omega_{\mathrm{rig}}]) \times H^1(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{R})/H^1(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{Z}) = \mathbb{Z}_2 \times \{0\}.$$

More precisely, the first factor parametrizes admissible quantum bundles and the second factor parametrizes quantum connections. □

In the following, we will specify the two possible trivial and non-trivial cases by the superscripts + or − only when it is required by the context.

Degenerate case. Next, we analyse the degenerate case, following the same lines of the non-degenerate case.

Proposition 3.5. *We have countably many equivalence classes of complex line bundles with basis $\boldsymbol{E}_{\mathrm{rig}}$ and just one equivalence class of quantum bundles. Namely, this is the trivial one.*

Proof. The second cohomology group of $\boldsymbol{E}_{\mathrm{rig}}$ is

$$H^2(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{Z}) \simeq H^2(\boldsymbol{S}_{\mathrm{rig}}, \mathbb{Z}) \simeq H^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}$$
$$H^2(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{R}) \simeq H^2(\boldsymbol{S}_{\mathrm{rig}}, \mathbb{R}) \simeq H^2(S^2, \mathbb{R}) \simeq \mathbb{R}.$$

Then, according to proposition 1.1, the equivalence classes of complex vector bundles are in bijection with $H^2(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{Z}) \simeq \mathbb{Z}$ and the equivalence classes of quantum bundles are in bijection with $(i^2)^{-1}([\Omega]) = (i^2)^{-1}(0) = \{0\}$. □

Theorem 3.6. *Let $\boldsymbol{E}_{\mathrm{rig}}$ be degenerate. Then, the only quantum structure is of the type $(\boldsymbol{Q}_{\mathrm{rig}}, \text{Ч}^{\uparrow}{}_{\mathrm{rig}})$, with*

$$\text{Ч}^{\uparrow}{}_{\mathrm{rig}} = \chi^{\uparrow}{}_{\mathrm{rig}} + \mathrm{i}\, A^{\uparrow}{}_{\mathrm{rig}} \otimes \mathbb{I}^{\uparrow},$$

where $\chi^{\uparrow}{}_{\mathrm{rig}}$ is the pullback of χ_{rig} and $A^{\uparrow}{}_{\mathrm{rig}}$ is a global horizontal potential for Ω_{rig}.

Proof. According to proposition 1.1, inequivalent quantum structures are in bijection with the set

$$(i^2)^{-1}([\Omega_{\mathrm{rig}}]) \times H^1(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{R})/H^1(\boldsymbol{E}_{\mathrm{rig}}, \mathbb{Z}) = \{0\} \times \{0\}.$$

More precisely, the first factor parametrizes admissible quantum bundles and the second factor parametrizes quantum connections. □

Distinguished representatives. In both non-degenerate and degenerate cases, the following facts hold.

Proposition 3.7. *Let us consider a global observer $o : \boldsymbol{E}_{\mathrm{rig}} \to J_1\boldsymbol{E}_{\mathrm{rig}}$.*

The form $-\mathcal{K}_{\mathrm{rig}}[o] + \mathcal{Q}_{\mathrm{rig}}[o]$ turns out to be a global horizontal potential for $\Omega^{\natural}$.

Then, in the particular case when $F = 0$, we can choose a representative of the quantum structure $(\boldsymbol{Q}_{\mathrm{rig}}, \text{Ч}^{\uparrow}{}_{\mathrm{rig}})$ in each equivalence class, such that $A_{\mathrm{rig}}[o] = 0$.

Hence, the quantum differential turns out to be just the covariant differential $\nabla[\chi_{\mathrm{rig}}]$ associated with the flat connection(s) χ_{rig} and the observed quantum Laplacian turns out to be just the (space-like) scaled Bochner Laplacian $\Delta[\chi_{\mathrm{rig}}, g_{\mathrm{rig}}]$ of the quantum bundle induced by the flat connection(s) χ_{rig} and the (space-like) metric g_{rig}.

Proposition 3.8. *The Hermitian quantum bundle can be written, up to an equivalence, as the fibred complex tensor product over $\boldsymbol{T}$*

$$\boldsymbol{Q}_{\mathrm{rig}} = \boldsymbol{Q}_{\mathrm{cen}} \otimes \boldsymbol{Q}_{\mathrm{rot}},$$

where $\boldsymbol{Q}_{\mathrm{cen}} \to \boldsymbol{E}_{\mathrm{cen}}$ is a Hermitian (trivial) quantum bundle over $\boldsymbol{E}_{\mathrm{cen}}$ and $\boldsymbol{Q}_{\mathrm{rot}} \to \boldsymbol{E}_{\mathrm{rot}}$ is a Hermitian quantum bundle over $\boldsymbol{E}_{\mathrm{rot}}$.

Accordingly, each quantum section Ψ_{rig} can be written as a finite sum of tensor products of the type $\Psi_{\mathrm{cen}} \otimes \Psi_{\mathrm{rot}}$, where $\Psi_{\mathrm{cen}} : \boldsymbol{E}_{\mathrm{rig}} \to \boldsymbol{Q}_{\mathrm{cen}}$ and $\Psi_{\mathrm{rot}} : \boldsymbol{E}_{\mathrm{rig}} \to \boldsymbol{Q}_{\mathrm{rot}}$.

3.2. Quantum dynamics

Now, we apply the machinery of 'covariant quantum mechanics' to each one of the above three possible choices of quantum structures.

We will not repeat the whole procedure, but only sketch the main differences between the one-body case and the rigid-body case. As one can expect, the most remarkable facts are due to the splitting $\boldsymbol{E}_{\mathrm{rig}} = \boldsymbol{E}_{\mathrm{cen}} \times \boldsymbol{S}_{\mathrm{rot}}$.

Thus, let us consider the quantum bundle $\boldsymbol{Q}_{\mathrm{rig}} \to \boldsymbol{E}_{\mathrm{rig}}$ and the phase quantum connection $Ч^{\uparrow}{}_{\mathrm{rig}}$. We have the splitting into coupled translational and rotational components

$$A^{\uparrow}{}_{\mathrm{rig}} = A^{\uparrow}{}_{\mathrm{cen}} + A^{\uparrow}{}_{\mathrm{rot}}, \quad \text{with} \quad A^{\uparrow}{}_{\mathrm{cen}} : J_1\boldsymbol{E}_{\mathrm{rig}} \to T^*\boldsymbol{E}_{\mathrm{cen}}, \quad A^{\uparrow}{}_{\mathrm{rot}} : J_1\boldsymbol{E}_{\mathrm{rig}} \to T^*\boldsymbol{S}_{\mathrm{rot}}.$$

The above splitting yields several other splittings. In particular, we can write

$$\overset{o}{\Delta}_{\mathrm{rig}} = \overset{o}{\Delta}_{\mathrm{cen}} + \overset{o}{\Delta}_{\mathrm{rot}} \qquad \text{and} \qquad \mathrm{S}_{\mathrm{rig}0} = \mathrm{S}_{\mathrm{cen}0} + \bar{\mathrm{S}}_{\mathrm{rot}0},$$

where

$$\overset{o}{\Delta}_{\mathrm{cen}}\psi = G_{\mathrm{cen}}{}^{ij}_{0}(\partial_i - \mathrm{i}A_{\mathrm{cen}\,i})(\partial_j - \mathrm{i}\,A_{\mathrm{cen}\,j}) + \frac{\partial_i\big(G_{\mathrm{cen}}{}^{ij}_{0}\sqrt{|g_{\mathrm{cen}}|}\big)}{\sqrt{|g_{\mathrm{cen}}|}}(\partial_j - \mathrm{i}\,A_{\mathrm{cen}\,j})\psi,$$

$$\overset{o}{\Delta}_{\mathrm{rot}}\psi = G_{\mathrm{rot}}{}^{\alpha\beta}_{0}(\partial_\alpha - \mathrm{i}\,A_{\mathrm{rot}\,\alpha})(\partial_\beta - \mathrm{i}\,A_{\mathrm{rot}\,\beta}) + \frac{\partial_\alpha\big(G_{\mathrm{rot}}{}^{\alpha\beta}_{0}\sqrt{|g_{\mathrm{rot}}|}\big)}{\sqrt{|g_{\mathrm{rot}}|}}(\partial_\beta - \mathrm{i}\,A_{\mathrm{rot}\,\beta})\psi,$$

$$\mathrm{S}_{\mathrm{cen}\,0}\psi = \left(\partial_0 - \mathrm{i}\,A_{\mathrm{cen}\,0} + \frac{1}{2}\frac{\partial_0\sqrt{|g_{\mathrm{cen}}|}}{\sqrt{|g_{\mathrm{cen}}|}} + \frac{1}{2}\overset{o}{\Delta}_{\mathrm{cen}\,0}\right)\psi,$$

$$\bar{\mathrm{S}}_{\mathrm{rot}\,0}\psi = \left(\frac{1}{2}\frac{\partial_0\sqrt{|g_{\mathrm{rot}}|}}{\sqrt{|g_{\mathrm{rot}}|}} + \frac{1}{2}(\overset{o}{\Delta}_{\mathrm{rot}\,0} + k\rho_{\mathrm{rot}\,0})\right)\psi.$$

The Lie algebra of special phase functions $\mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{rig}}, \mathbb{R})$ has two remarkable subspaces, namely $\mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{cen}}, \mathbb{R})$ and $\mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{rot}}, \mathbb{R})$. These subspaces turn out to be subalgebras in the case when $\mathcal{F}_{\mathrm{rig}}$ is decoupled with respect to $\boldsymbol{E}_{\mathrm{cen}}$ and $\boldsymbol{S}_{\mathrm{rot}}$. In this case, we have 'translational' and 'rotational' observables.

If $f_{\mathrm{cen}} \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{cen}}, \mathbb{R})$ and $f_{\mathrm{rot}} \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{rot}}, \mathbb{R})$, then we have the coordinate expressions

$$f_{\mathrm{cen}} = f^0_{\mathrm{cen}}\tfrac{1}{2}\,G_{\mathrm{cen}}{}^{0}_{ij}x^i_0x^j_0 + f^i_{\mathrm{cen}}G_{\mathrm{cen}}{}^{0}_{ij}x^j_0 + \check{f}_{\mathrm{cen}},$$

$$f_{\mathrm{rot}} = f^0_{\mathrm{rot}}\tfrac{1}{2}\,G_{\mathrm{rot}}{}^{0}_{\alpha\beta}x^\alpha_0x^\beta_0 + f^\alpha_{\mathrm{rot}}G_{\mathrm{rot}}{}^{0}_{\alpha\beta}x^\beta_0 + \check{f}_{\mathrm{rot}}.$$

The associated quantum operators are

$$\widehat{f}_{\mathrm{cen}}\psi = \big(\check{f}_{\mathrm{cen}} - \mathrm{i}\,\tfrac{1}{2}\,\partial_j f^j_{\mathrm{cen}} - \mathrm{i}\,f^j_{\mathrm{cen}}(\partial_j - \mathrm{i}\,A_{\mathrm{cen}\,j}) - \tfrac{1}{2}\,f^0_{\mathrm{cen}}\overset{o}{\Delta}_{\mathrm{cen}\,0}\big)\psi,$$

$$\widehat{f}_{\mathrm{rot}}\psi = \big(\check{f}_{\mathrm{rot}} - \mathrm{i}\,\tfrac{1}{2}\,\partial_\alpha f^\alpha_{\mathrm{rot}} - \mathrm{i}\,f^\alpha_{\mathrm{rot}}(\partial_\alpha - \mathrm{i}\,A_{\mathrm{rot}\,\alpha}) - \tfrac{1}{2}\,f^0_{\mathrm{rot}}(\overset{o}{\Delta}_{\mathrm{rot}\,0} + k\rho_{\mathrm{rot}\,0})\big)\psi.$$

In particular, we have the following special phase functions:

$$x^0, x^i_{\mathrm{cen}} \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{cen}}, \mathbb{R}), \qquad x^0, x^\alpha_{\mathrm{rot}} \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{rot}}, \mathbb{R}),$$

$$\mathcal{P}_{\mathrm{cen}\,j}, \|\mathcal{P}_{\mathrm{cen}}\|^2 \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{cen}}, \mathbb{R}), \qquad \mathcal{P}_{\mathrm{rot}\,\alpha}, \|\mathcal{P}_{\mathrm{rot}}\|^2 \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{rot}}, \mathbb{R}),$$

$$\mathcal{H}_{\mathrm{cen}\,0} \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{cen}}, \mathbb{R}), \qquad \mathcal{H}_{\mathrm{rot}\,0} \in \mathrm{spec}(J_1\boldsymbol{E}_{\mathrm{rot}}, \mathbb{R}).$$

and the associated quantum operators

$$\widehat{x^i_{\mathrm{cen}}}\psi = x^i\psi, \qquad \widehat{\mathcal{P}_{\mathrm{cen}\,j}}\psi = -\mathrm{i}\left(\partial_j + \frac{1}{2}\frac{\partial_j\sqrt{|g_{\mathrm{cen}}|}}{\sqrt{|g_{\mathrm{cen}}|}}\right)\psi,$$

$$\widehat{\mathcal{H}_{\mathrm{cen}\,0}}\psi = \left(-\frac{1}{2}\overset{o}{\Delta}_{\mathrm{cen}\,0} - A_{\mathrm{cen}\,0}\right)\psi,$$

$$\widehat{x^\alpha_{\mathrm{rot}}}\psi = x^\alpha_{\mathrm{rot}}\psi, \qquad \widehat{\mathcal{P}_{\mathrm{rot}\,\alpha}}\psi = -\mathrm{i}\left(\partial_\alpha + \frac{1}{2}\frac{\partial_\alpha\sqrt{|g_{\mathrm{rot}}|}}{\sqrt{|g_{\mathrm{rot}}|}}\right)\psi,$$
$$\widehat{\mathcal{H}_{\mathrm{rot}\,0}}\psi = \left(-\frac{1}{2}\overset{o}{\Delta}_{\mathrm{rot}\,0} + k\rho_{\mathrm{rot}\,0}\right)\psi, \tag{12}$$

and

$$\widehat{\|\mathcal{P}_{\mathrm{cen}}\|^2_0}\psi = \left(G^{ij}_{\mathrm{cen}\,0}A_{\mathrm{cen}\,i}A_{\mathrm{cen}\,j} - \mathrm{i}\left(\partial_h A^h_{\mathrm{cen}} + 2A^h_{\mathrm{cen}}(\partial_h - \mathrm{i}\,A_{\mathrm{cen}\,h})\right) - \overset{o}{\Delta}_{\mathrm{cen}\,0}\right)\psi$$
$$\widehat{\|\mathcal{P}_{\mathrm{rot}}\|^2_0}\psi = \left(G^{\alpha\beta}_{\mathrm{rot}\,0}A_{\mathrm{rot}\,\alpha}A_{\mathrm{rot}\,\beta} - \mathrm{i}\left(\partial_\alpha A^\alpha_{\mathrm{rot}} + 2A^\alpha_{\mathrm{rot}}(\partial_\alpha - \mathrm{i}\,A_{\mathrm{rot}\,\alpha})\right) - \overset{o}{\Delta}_{\mathrm{rot}\,0} - k\rho_{\mathrm{rot}\,0}\right)\psi.$$

4. Rotational quantum spectra

4.1. Angular momentum in the free case

Here we analyse the implementation of angular momentum for a rigid body in the framework of covariant quantum mechanics. We start by recalling the relevant facts concerning angular momentum in covariant classical mechanics. In this case it is well known that angular momentum appears as a conserved quantity of systems which are invariant under rotations. More precisely, in these systems the angular momentum can be interpreted as a momentum map for the action of the rotation group (see [49] for further details on symmetries in covariant classical mechanics). This momentum map takes values in the special functions; hence we associate with every element of the Lie algebra of the rotation group a quantum operator, and we get in this way a Lie algebra representation whose Casimir is the square angular momentum operator.

We consider the following group actions:

$$SO(\boldsymbol{S}, g) \times (\mathbb{T} \times \boldsymbol{S}_{\mathrm{rot}}) \to \mathbb{T} \times \boldsymbol{S}_{\mathrm{rot}} : (A, (\tau, r)) \mapsto (\tau, A(r)).$$

We would like to find the invariance of the dynamical structures with respect to the above action. To this aim, we choose a global potential $A^{\uparrow}$. We observe that $A^{\uparrow}$ splits into the sum $A^{\uparrow} = A^{\uparrow}{}_{\mathrm{cen}} + A^{\uparrow}{}_{\mathrm{rot}}$ in an obvious way.

Proposition 4.1. *The group $SO(\boldsymbol{S}, g)$ is a group of symmetries of the potential $A^{\uparrow}{}_{\mathrm{rot}}$. Moreover, the momentum map induced by the action of $SO(\boldsymbol{S}, g)$ is just the total angular momentum with respect to the centre of mass.*

Proof. In fact, $A^{\uparrow}{}_{\mathrm{rot}}$ reduces to the kinetic energy of particles with respect to the centre of mass. It is not difficult to prove that it is invariant with respect to orthogonal transformations (see [12]). We have the momentum map

$$J : \mathfrak{so}(\boldsymbol{S}, g) \to C^\infty(J_1(\boldsymbol{T} \times \boldsymbol{S}_{\mathrm{rot}})) : \omega \mapsto J(\omega) \equiv \omega^* \circ \mathcal{P}_{\mathrm{rot}}.$$

Here, $\omega^* : \boldsymbol{S}_{\mathrm{rot}} \to T\boldsymbol{S}_{\mathrm{rot}} : r \mapsto \omega(r)$; moreover, $J_1(\boldsymbol{T} \times \boldsymbol{S}_{\mathrm{rot}}) = \boldsymbol{T} \times \mathbb{T}^* \otimes T\boldsymbol{S}_{\mathrm{rot}}$. It is easy to show that $\omega^* \circ \mathcal{P}_{\mathrm{rot}}(v) = G_{\mathrm{rot}}(\omega(r), v)$, where $v \in \mathbb{T}^* \otimes T\boldsymbol{S}_{\mathrm{rot}}$. We have the coordinate expression $J(\omega) = (G_{\mathrm{rot}})^0_{\alpha\beta} x^\beta_0 (\omega^*)^\alpha$.

The Hodge star isomorphism yields a natural Lie algebra isomorphism $\mathfrak{so}(\boldsymbol{S}, g) \simeq \mathbb{L}^{-1} \otimes \boldsymbol{S}$ sending the Lie bracket of $\mathfrak{so}(\boldsymbol{S}, g)$ into the cross product. In this way, if $\omega \in \mathfrak{so}(\boldsymbol{S}, g)$ and $\bar{\omega} \in \mathbb{L}^{-1} \otimes \boldsymbol{S}$ is the corresponding element, then we can equivalently write

$$J : \mathbb{L}^{-1} \otimes \boldsymbol{S} \to C^\infty(J_1(\boldsymbol{T} \times \boldsymbol{S}_{\mathrm{rot}})) : \bar{\omega} \mapsto J(\bar{\omega}) \equiv G_{\mathrm{rel}}(r \times v, \omega).$$

This proves the last part of the statement. □

The map J takes values into the space of special functions since $J(\omega)$ is a linear function of velocities for each $\omega \in \mathfrak{so}(\boldsymbol{S}, g)$. Hence, it makes sense to consider the lift of $J(\omega)$ to a quantum operator.

More precisely, by a composition of the momentum map with the lift of quantum functions to quantum operators we obtain the following representation of the Lie algebra $\mathfrak{so}(\boldsymbol{S}, g)$:

$$\hat{J} : \mathfrak{so}(\boldsymbol{S}, g) \to \mathrm{Op}(\hat{\boldsymbol{Q}}, \hat{\boldsymbol{Q}}) : \omega \mapsto \widehat{J(\omega)},$$

If we consider a global observer $o : \boldsymbol{E}_{\mathrm{rig}} \to J_1\boldsymbol{E}_{\mathrm{rig}}$ then we have

$$\widehat{J(\omega)} = \mathfrak{i}\left(X[J(\omega)] \lrcorner \nabla[o] + \tfrac{1}{2}\, \mathrm{div}_\eta\, X[J(\omega)]\right) + J(\omega)[o]$$

but $J(\omega)[o] = 0$ and $\mathrm{div}_\eta\, X[J(\omega)] = 0$ since G^0_{rot} is a left invariant metric and $X[J(\omega)]$ is the fundamental vector field associated with $\omega \in \mathfrak{so}(\boldsymbol{S}, g)$. Therefore

$$\widehat{J(\omega)} = \mathfrak{i}\, X[J(\omega)] \lrcorner \nabla[o].$$

Let us consider a basis $\{\omega_1, \omega_2, \omega_3\}$ of the Lie algebra $\mathfrak{so}(\boldsymbol{S}, g)$ which be orthonormal with respect to the metric σ_{rot} (recall that σ_{rot} is isometric to $-\frac{1}{2}\, k_3$, where k_3 is the Killing metric of $SO(3)$).

Definition 4.2. *The square angular momentum operator* $\hat{J}^2$ *is* $\hbar^2 C$, *where C is the Casimir of the Lie algebra representation* $\hat{J} : \mathfrak{so}(\boldsymbol{S}, g) \to \mathrm{Op}(\hat{\boldsymbol{Q}}, \hat{\boldsymbol{Q}})$, *thus*

$$\hat{J}^2 = \hbar^2 C = \hbar^2\left(\widehat{J(\omega_1)} \circ \widehat{J(\omega_1)} + \widehat{J(\omega_2)} \circ \widehat{J(\omega_2)} + \widehat{J(\omega_3)} \circ \widehat{J(\omega_3)}\right).$$

Note 4.3. The differential operator C is exactly the pullback to $\boldsymbol{Q}$ of the Bochner Laplacian $\Delta[\chi_{\mathrm{rot}}]$ of the line bundle $\boldsymbol{Q}_{\mathrm{rot}} \to \boldsymbol{S}_{\mathrm{rot}}$ with respect to the connection χ_{rot} of $\boldsymbol{Q}_{\mathrm{rot}}$ and the Riemannian metric σ_{rot} of $\boldsymbol{S}_{\mathrm{rot}}$.

4.2. Energy in the free case

In this section we assume that the electromagnetic field vanishes. In such a case, the Schrödinger equation splits into the two decoupled Schrödinger equations for the centre of mass and rotations. Clearly, the first one is trivial. So, we concentrate our attention just on the rotational Schrödinger equation.

We evaluate the spectra of rotational Hamiltonian for both non-degenerate (for trivial and non-trivial quantum bundles) and degenerate cases.

Let us set $F = 0$, and refer to a global inertial observer $o : \boldsymbol{E}_{\mathrm{rig}} \to J_1\boldsymbol{E}_{\mathrm{rig}}$ and to a representative of the quantum structure $(\boldsymbol{Q}_{\mathrm{rig}}, \mathrm{Ч}^{\uparrow}{}_{\mathrm{rig}})$ in the unique equivalence class, such that $A_{\mathrm{rig}}[o] = 0$, according to proposition 3.7. So, we consider the quantum bundle $\boldsymbol{Q}_{\mathrm{rot}} \to \boldsymbol{S}_{\mathrm{rot}}$, which may be trivial or not, and the associated sectional quantum bundle $\widehat{\boldsymbol{Q}}_{\mathrm{rot}} \to \boldsymbol{T}$.

Let us consider the quantum Hamiltonian operator

$$\widehat{\mathcal{H}_{\mathrm{rot}\,0}} = -\tfrac{1}{2}\,(\overset{o}{\Delta}_{\mathrm{rot}\,0} + k\rho_{\mathrm{rot}\,0}),$$

where, according to our choices,

$$\overset{o}{\Delta}_{\mathrm{rot}\,0} = \Delta\big[\chi_{\mathrm{rot}}, G^0_{\mathrm{rot}}\big] = \Delta\left[\chi_{\mathrm{rot}}, \frac{m}{\hbar_0}\, g_{\mathrm{rot}}\right]$$

turns out to be just the (unscaled) metric Laplacian associated with the flat connection χ and the Riemannian metric G^0_{rot}. We stress that $\Delta\big[G^0_{\mathrm{rot}}\big]$ does not depend on the choice of an observer, as $\boldsymbol{S}_{\mathrm{rot}}$ is space-like, while $\overset{o}{\Delta}_{\mathrm{rot}\,0}$ depends on the choice of the observer o, which yields $A_{\mathrm{rig}}[o] = 0$. Thus, the above equality holds just for that observer.

Lemma 4.4 [5, p 145.]. *Let $(\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian covering. Then, the eigenfunctions of the Laplacian $\Delta[g]$ are the projections on M of the projectable eigenfunctions of the Laplacian $\Delta[\tilde{g}]$. Moreover, we have* $\operatorname{Spec} \Delta[g] \subset \operatorname{Spec} \Delta[\tilde{g}]$.

Lemma 4.5. *Let $(\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian covering. Let $Q \to M$ be a complex line bundle obtained as quotient of the trivial line bundle $\tilde{Q} := \tilde{M} \times \mathbb{C}$, with respect to the equivalence relation induced by the covering. Moreover, let us suppose that the bundle $Q \to M$ is equipped with a flat connection χ obtained as quotient from the trivial flat connection $\tilde{\chi}$ of the bundle $\tilde{Q} \to \tilde{M}$. Let us consider the Bochner Laplacians $\Delta[\tilde{\chi}, \tilde{g}]$ and $\Delta[\chi, g]$ of $\tilde{Q}$ and Q, respectively. Then, the eigensections of the Laplacian $\Delta[\chi, g]$ are the projections to sections of $Q \to M$ of the projectable eigensections of the Laplacian $\Delta[\tilde{\chi}, \tilde{g}]$. Moreover, the corresponding eigenvalues are the same.*

Lemma 4.6 [5, pp 159, 160.]. *Let $(S^n, \tilde{g}) \subset (\mathbb{R}^{n+1}, \mathrm{g})$ be the standard sphere. Then, we have* $\operatorname{Spec} \Delta[\tilde{g}] = \{\lambda_d = -d(d+n-1) \mid d \geqslant 0\}$. *Moreover, the eigenspace $\tilde{\mathcal{H}}_d$ associated with λ_d consists of the restrictions to S^n of harmonic homogeneous polynomials of degree d of $\mathbb{R}^{n+1}$. We have* $\dim \tilde{\mathcal{H}}_d = \binom{n+d-2}{d}\left(\frac{2d+n-1}{n-1}\right)$.

By lemm 4.5 we can identify the Casimir operator C, which acts on sections of the line bundle $Q_{\mathrm{rot}} \to S_{\mathrm{rot}}$, with an operator $\tilde{C}$ acting on functions on S^3. One can prove (see [53, lemma 7]) that $\tilde{C} = \frac{1}{4}\Delta[\tilde{g}]$, where $\Delta[\tilde{g}]$ is the Laplacian of the standard Riemannian metric of S^3. Thanks to lemm 4.6 we have

Theorem 4.7. *The spectrum of $\hat{J}^2$ is*

$$\operatorname{Spec}(\hat{J}^2) = \{\hbar j(j+1)\},$$

where $j \in \mathbb{N}$ in the trivial case and $j \in \frac{1}{2}\mathbb{N}$ in the non-trivial case.

The complex multiplicity of the eigenvalue $j(j+1)$ is $(2j+1)^2$.

The eigensections with eigenvalue $j(j+1)$ are the projections to $Q_{\mathrm{rot}} \to S_{\mathrm{rot}}$ of the restrictions to S^3 of homogeneous harmonic complex polynomials in $\mathbb{R}^4$ of degree $2j$ in the trivial case and of degree $2j+1$ in the non-trivial case.

Theorem 4.8. *Spherical, non-degenerate case. The spectrum of $\widehat{\mathcal{H}_{\mathrm{rot}\,0}}$ is*

$$\operatorname{Spec}(\widehat{\mathcal{H}_{\mathrm{rot}\,0}}) = \left\{ E_j = \frac{\hbar_0}{2I}\, j(j+1) + k\frac{3\hbar_0}{2I} \right\},$$

where $j \in \mathbb{N}$ in the trivial case and $j \in \frac{1}{2}\mathbb{N}$ in the non-trivial case.

The complex multiplicity of the eigenvalue E_j is $(2j+1)^2$.

The eigensections of E_j are the projections to $Q_{\mathrm{rot}} \to S_{\mathrm{rot}}$ of the restrictions to S^3 of homogeneous harmonic complex polynomials in $\mathbb{R}^4$ of degree $2j$ in the trivial case and of degree $2j+1$ in the non-trivial case.

Proof. We restrict our attention to $-\frac{1}{2}\overset{o}{\Delta}_{\mathrm{rot}\,0}$, since the contribution of the scalar curvature is obvious.

In virtue of the computations of subsection 2.3.2 we have an isometry $S_{\mathrm{rot}} \to SO(3)$, with respect to the metrics $G_{\mathrm{rot}\,0} = \frac{m}{\hbar_0}\, g_{\mathrm{rot}} = \frac{m}{\hbar_0}\frac{I}{m}\,\sigma_{\mathrm{rot}}$ and $-\frac{1}{2}\frac{I}{\hbar_0}\,k_3$, respectively. Hence, the standard two-fold Riemannian covering $S^3 \to SO(3)$ yields a two-fold Riemannian covering $S^3 \to S_{\mathrm{rot}}$, with respect to the metrics $-\frac{1}{2}\frac{I}{\hbar_0}\, g_3$ and $G_{\mathrm{rot}\,0} = \frac{m}{\hbar_0}\, g_{\mathrm{rot}}$, respectively, where g_3 is the metric induced on S^3 by the Killing metric of $SU(2)$ via the natural identification $S^3 \simeq SU(2)$.

We recall that $\boldsymbol{Q}_{\text{rot}}$ can be obtained from the trivial bundle $S^3 \times \mathbb{C} \to S^3$ by a quotient (see lemma 3.2).

If $\tilde{g}$ is the standard metric of the sphere then one has $-\frac{1}{2}\, g_3 = 4\tilde{g}$. Therefore, the theorem follows from lemmas 4.4, 4.5 and 4.6, by taking into account that the eigenspace $\tilde{\mathcal{H}}_d$ is projectable on $\boldsymbol{Q}_{\text{rot}}$ if d is even or odd in the trivial case or in the non-trivial case, respectively. □

Corollary 4.9. *Spherical, non-degenerate case. The eigensections with eigenvalue E_j are eigensections of the square angular momentum operator with eigenvalue $\hbar_0^2 j(j+1)$.*

Lemma 4.10. *Let us consider an axially symmetric rigid body. Let $(X_1, X_2, X_3) \subset \boldsymbol{V}_{\text{ang}}$ be an orthonormal basis, with respect to g, where X_1 has the direction of the symmetry axis.*

The corresponding basis of $T\boldsymbol{S}_{\text{rot}}$ (denoted by the same symbol) turns out to be left invariant and such that $G^0_{\text{rot}}(X_i, X_i) = \frac{I_i}{\hbar_0}$.

Then, we obtain

$$\overset{o}{\Delta}_{\text{rot}\,0} = \Delta\big[\chi_{\text{rot}}, G^0_{\text{rot}}\big] = \frac{\hbar_0}{2I} C + \left(\frac{\hbar_0}{I_3} - \frac{\hbar_0}{I}\right)(X_3 \lrcorner \nabla[\chi_{\text{rot}}] \circ X_3 \lrcorner \nabla[\chi_{\text{rot}}]),$$

where $X_3 \lrcorner \nabla[\chi_{\text{rot}}]$ is regarded in a natural way as a differential operator acting on sections of $\boldsymbol{Q}_{\text{rot}}$.

Proof. We can write

$$\Delta\big[\chi_{\text{rot}}, G^0_{\text{rot}}\big] = \frac{\hbar}{I_1}(X_1 \lrcorner \nabla[\chi_{\text{rot}}])^2 + \frac{\hbar}{I_2}\big(X_2 \lrcorner \nabla[\chi_{\text{rot}}]\big)^2 + \frac{\hbar}{I_3}(X_3 \lrcorner \nabla[\chi_{\text{rot}}])^2.$$ □

We say that an eigenvalue depending on two parameters has *arithmetical degeneracy* if it can be obtained from different pairs of values of the parameters.

We note that complex polynomials in $\mathbb{R}^4$ can be regarded as complex polynomials in the variables $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ [53, p 169].

Theorem 4.11. *Symmetric non-degenerate case. The spectrum of $\widehat{\mathcal{H}_{\text{rot}\,0}}$ is*

$$\text{Spec}(\widehat{\mathcal{H}_{\text{rot}\,0}}) = \left\{E_{j,l} = \frac{\hbar_0}{2I}\, j(j+1) + \frac{\hbar_0}{2}\left(\frac{1}{I_3} - \frac{1}{I}\right) l^2 + k\hbar_0\left(\frac{2}{I} - \frac{I_3}{2I^2}\right)\right\},$$

where $j \in \mathbb{N}$ and $l \in \mathbb{Z}$ in the trivial case and $2j \in \mathbb{N} \setminus 2\mathbb{N}$ and $2l \in \mathbb{Z} \setminus 2\mathbb{Z}$ in the non-trivial case.

In case that there is no arithmetical degeneracy, the multiplicity of the eigenvalue $E_{j,l}$ is $2(2j+1)$.

Eigensections of $E_{j,l}$ are the projections to $\boldsymbol{Q}_{\text{rot}} \to \boldsymbol{S}_{\text{rot}}$ of complex homogeneous harmonic polynomials in $\mathbb{R}^4$ of degree p in z_i and degree q in $\bar{z}_i$, with $p + q = 2j$, such that X_3 has eigenvalue l on them.

Proof. The result can be obtained in the same way as theorem 4.8, by using the above Lemma and the fact that the operators $\Delta\big[\chi_{\text{rot}}, G^0_{\text{rot}}\big]$ and $(X_1 \lrcorner \nabla[\chi_{\text{rot}}])^2$ commute [53].

Of course, the eigenvalues of $(X_1 \lrcorner \nabla[\chi_{\text{rot}}])^2$ are square integers and square half-integers on S^3. □

Corollary 4.12. *Symmetric non-degenerate case. The eigensections with eigenvalue $E_{j,l}$ are eigensections of the square angular momentum operator with eigenvalue $\hbar_0^2 j(j+1)$.*

Arithmetical degeneracy can occur if $I_3/(I_3 - I) \in \mathbb{Q}$. In this case, we could have $E_{j,l} = E_{j',l'}$ for some $j \neq j'$ or $l \neq l'$. See [53] for more details about the computation and the multiplicity of eigenvalues and eigensections.

Note 4.13. Let us consider the asymmetric non-degenerate case.

There is no general solution for the spectral problem, but just a general method by which finding the solution in each case. Namely, it is possible to restrict the Laplace operator to $(p + q + 1)$-dimensional subspaces $H^{p,q}$ of harmonic complex polynomials of $\mathbb{R}^4$ which are of degree p in z_i and degree q in $\bar{z}_i$, restricted to S^3.

The eigenvalue problem is solved by finding the root of the characteristic polynomial, which is of degree $p + q + 1$. Of course, the complexity of this problem increases with p and q. See [53] for more details.

For the sake of completeness, we also mention the following more standard result [5], which follows directly from lemma 4.6.

Theorem 4.14. *Degenerate case. The spectrum of* $\widehat{\mathcal{H}_{\text{rot }0}}$ *is*

$$\operatorname{Spec}(\widehat{\mathcal{H}_{\text{rot }0}}) = \left\{ E_j = \frac{\hbar_0}{2I}\, j(j+1) + k\hbar_0 \frac{2}{I} \right\},$$

where $j \in \mathbb{N}$.

The multiplicity of the eigenvalue E_j *is* $(2j+1)^2$.

Eigensections of E_j *are the harmonic complex polynomials in* $\mathbb{R}^3$ *restricted to* S^2 *of degree* $2j$.

In this case, the system is again invariant under rotations and admits a momentum map which can be interpreted as the angular momentum. Proceeding in a similar way as above we get

Corollary 4.15. *Degenerate case. The eigensections with eigenvalue* E_j *are eigensections of the square angular momentum operator with eigenvalue* $\hbar_0^2 j(j+1)$.

4.3. Spectra with electromagnetic field

If the electromagnetic field does not vanish the computations of the spectra might become quite hard. However, specific problems can be faced.

Here, we sketch typical evaluations, with reference to the literature, showing how they can be rephrased in our framework. Indeed, our non-trivial bundle structure opens a possible geometric interpretation of the 'two-valued' wavefunctions, which seems to be closely related to spin.

Example 4.16. (*Stark effect.*) The energy spectrum of a charged rigid body rotating in a constant external electric, or magnetic field can be computed in our framework along the lines of the previous section.

Let us consider an inertial observer o, and assume that F is a constant electric field with respect to o, i.e., $\vec{E}[o] \in \mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2} \otimes \boldsymbol{S}$ and $\vec{B} := 0$. In order to write down the energy operator we have to evaluate the electric potential of $\mathcal{F}_{\text{rig}}$. This can be done along the lines sketched in the discussion of equations (3) and (11). Namely, we have

$$\mathcal{F}_{\mathrm{cen}}(e_{\mathrm{rig}}; v_{\mathrm{rig}}, w_{\mathrm{rig}}) = \sum_i \frac{q_i}{m} F_i(e_i; v_{\mathrm{cen}\,i}, w_{\mathrm{cen}\,i})$$
$$\mathcal{F}_{\mathrm{rot}}(e_{\mathrm{rig}}; v_{\mathrm{rig}}, w_{\mathrm{rig}}) = \sum_i \frac{q_i}{m} F_i(e_i; \omega \times r_i, \psi \times r_i)$$
$$\mathcal{F}_{\mathrm{cenrot}}(e_{\mathrm{rig}}; v_{\mathrm{rig}}, w_{\mathrm{rig}}) = \sum_i \frac{q_i}{m} F_i(e_i; v_{\mathrm{cen}\,i}, \psi \times r_i) + \sum_i \frac{q_i}{m} F_i(e_i; \omega \times r_i, w_{\mathrm{cen}\,i}),$$

for each

$$(e_{\mathrm{rig}}, v_{\mathrm{rig}}) = (e_{\mathrm{cen}}, r_1, \ldots, r_n; v_{\mathrm{cen}}, \omega \times r_1, \ldots, \omega \times r_n) \in T\boldsymbol{E}_{\mathrm{rig}},$$
$$(e_{\mathrm{rig}}, w_{\mathrm{rig}}) = (e_{\mathrm{cen}}, r_1, \ldots, r_n; w_{\mathrm{cen}}, \psi \times r_1, \ldots, \psi \times r_n) \in T\boldsymbol{E}_{\mathrm{rig}}.$$

Now, we use the observed splitting (1) together with our hypotheses on F to obtain

$$\mathcal{F}_{\mathrm{cen}}(e_{\mathrm{rig}}; v_{\mathrm{rig}}, w_{\mathrm{rig}}) = \frac{q}{m} \vec{E}[o] \cdot (\mathrm{d}t(w_{\mathrm{cen}})\vec{v}_{\mathrm{cen}}[o] - \mathrm{d}t(v_{\mathrm{cen}})\vec{w}_{\mathrm{cen}}[o])$$
$$\mathcal{F}_{\mathrm{rot}}(e_{\mathrm{rig}}; v_{\mathrm{rig}}, w_{\mathrm{rig}}) = 0$$
$$\mathcal{F}_{\mathrm{cenrot}}(e_{\mathrm{rig}}; v_{\mathrm{rig}}, w_{\mathrm{rig}}) = \frac{1}{m} \vec{E}[o] \cdot ((\mathrm{d}t(v_{\mathrm{cen}})\omega - \mathrm{d}t(w_{\mathrm{cen}})\psi) \times \vec{\mu}),$$

where $\vec{\mu} := \sum_i q_i r_i$ is the *dipole momentum* of the rigid body.

It is now easy to show that we obtain in coordinates the same energy operator of the literature, with the difference that it can operate on sections of the trivial or non-trivial quantum bundle. As an example, following [21] we assume that the rigid body is symmetric (i.e., a top) and that its dipole momentum is parallel to its main axis of symmetry. Then the potential $\mathcal{A}_{\mathrm{rig}} = A_{\mathrm{cen}\,0} u^0$ of $\mathcal{F}_{\mathrm{rig}}$ (of course, the vector part of $\mathcal{A}_{\mathrm{rig}}$ can be taken to be 0 because the magnetic field vanishes) is the sum of two terms $A_{\mathrm{cen}\,0} = A^c_{\mathrm{cen}\,0} + A^r_{\mathrm{cen}\,0}$ depending, respectively, on $\boldsymbol{E}_{\mathrm{cen}}$ and $\boldsymbol{S}_{\mathrm{rot}}$. This means that the energy operator decouples into a sum of two operators acting on sections of $\boldsymbol{Q}_{\mathrm{cen}}$, $\boldsymbol{Q}_{\mathrm{rot}}$, respectively. The second term of the potential is just

$$A^r_{\mathrm{cen}\,0} = \frac{1}{m} (\vec{E} \cdot \vec{\mu})_0.$$

The difference between the 'free' energy operator (12) and $A^r_{\mathrm{cen}\,0}$ yields exactly the same energy operator of [21] (up to the scalar curvature which, being constant, adds just an overall shift to the spectrum), so that the recursive equations that are used in that paper to compute the spectra (equation (14) on p 2801) hold also here. However, the solutions of the recursive equations depend on the spectrum of the free rigid body, which has been computed in theorem 4.11. Due to the fact that we also admit half-integer quantum numbers j (corresponding to sections of the non-trivial bundle), there will be additional solutions to the recursive equations parametrized by half-integers values of j, with corresponding wavefunctions as sections of the non-trivial bundle.

Example 4.17. (*Magnetic monopole*). Let us consider a rigid body with a fixed point at which a monopole is located. In this case we consider as electromagnetic pattern field the field generated by the monopole. In order to treat this case we have to change slightly the formalism developed above. The main difference with respect to the other examples of electromagnetic fields considered before lies in the fact that the existence of a fixed point reduces the configuration space to $\boldsymbol{E}_{\mathrm{rot}}$. Therefore, the centre-of-mass components do not appear and the magnetic field $\mathcal{F}_{\mathrm{rig}}$ reduces to $\mathcal{F}_{\mathrm{rot}}$ which is a 2-form on $\boldsymbol{S}_{\mathrm{rot}}$.

Proceeding as in the preceding example, one can easily compute the explicit expression of $\mathcal{F}_{\mathrm{rot}}$. In the non-degenerate case one obtains that $\mathcal{F}_{\mathrm{rot}}$ is a left invariant exact 2-form on

$\boldsymbol{S}_{\mathrm{rot}} \simeq SO(3)$, whereas in the degenerate case $\mathcal{F}_{\mathrm{rot}}$ is a constant times the Euclidean area element of $\boldsymbol{S}_{\mathrm{rot}} \simeq S^2$.

Following the steps of section 4.1, it is straightforward to see that in both cases the system is invariant under the action of $SO(3)$ and in the same way as there one obtains a representation of its lie algebra whose Casimir gives the corresponding square angular momentum operator $\hat{J}^2$.

In the non-degenerate case the cosymplectic form Ω is exact whereas in the degenerate case it defines a non-trivial cohomology class. However, both cases can be dealt with in a similar way by considering the fibration $S^3 \to \boldsymbol{S}_{\mathrm{rot}}$, which in the former case is the universal covering space and in the latter is the Hopf fibration. After lifting all the structures to S^3, the quantum bundle becomes trivial and the computations are performed, very much in the same way as we did for the free rigid body, by considering the new square angular momentum operator $\hat{J}^2$.

For instance, one finds that the spectrum of the energy operator in the case of a symmetric rigid body is

$$\mathrm{Spec}(\widehat{\mathcal{H}_{\mathrm{rot}\,0}}) = \left\{ E_{j,l} = \frac{\hbar_0}{2I}\, j(j+1) + \frac{\hbar_0}{2}\left(\frac{1}{I_1} - \frac{1}{I}\right) l^2 - \nu_0 \frac{\|\vec{q}\,\|}{I_3}\, l + \frac{\nu_0^2}{\hbar_0}\, \frac{\|\vec{q}\,\|^2}{2I_3} + k\rho_{\mathrm{rot}\,0} \right\},$$

where $j \geqslant 0, -j \leqslant l \leqslant j$, j, l are integers, for the trivial quantum bundle, or half–integers, for the non-trivial quantum bundle, ν is the magnetic charge of the monopole and $\vec{q}$ is the centre of charge of the rigid body defined by

$$\vec{q} = \sum_i q_i \frac{\xi_i}{\|\xi_i\|},$$

where ξ_i is the position vector of the ith particle with respect to the fixed point.

In the framework of geometric quantization one of us [53] has given an exact solution to the spectral problem of a rigid body in a magnetic monopole field.

5. Conclusions

There are some important questions which are not touched or left open in this paper. In this section we discuss the most important ones, in view of future research on the quantum rigid body.

Other approaches to rigid constraints. A different mathematical approach to rigid constraints is provided by considering a potential with suitable wells confining the constituent particles and by referring to the limit case when these forces freeze the distances between the particles [40]. In our paper we chose the 'ideal' approach of the classical rigidity constraint. This allowed us to achieve 'exact' computations of spectra, at least for the free rigid body and the rigid body under the action of a magnetic monopole field. So, in this sense, the rigid constraint yields a relatively simple analysis of the spectral problems. On the other hand, the problem of computing spectra of a multi-particle system under the action of bounding forces is mathematically much more complex and can be solved, e.g., by perturbative methods [40].

An approach which is a direct generalization of ours is the *pseudo-rigid body* [16, 55], where an extra degree of freedom is used to take into account dilations. However, also in this case one has the trivial and the non-trivial quantum structure, as it is easy to realize by a cohomological analysis like the one in subsection 3.1. This feature was not observed in [55].

Computing spectra in accelerated frames. In this paper we compute spectra only with respect to a fixed inertial observer. When the reference system is accelerated it is customary to add ad hoc terms to the standard Schrödinger operator in order to fit most spectral lines.

Our framework allows us to obtain the coordinate expression of Schrödinger operators with respect to$_o$accelerated frames. As an example, it is enough to compute the coordinate expression of $\Delta_{\mathrm{rot}\,0}$, $\rho_{\mathrm{rot}\,0}$ with respect to the chosen non-inertial coordinate system (12) to obtain the expression of the energy operator. The above quantities could also be expressed as the sum of their counterpart in an inertial frame plus non-inertial corrections.

Hence, in principle, it should be possible to compute the energy spectrum with respect to accelerated observers, again with the trivial and the non-trivial quantum structures. But again the problem would not admit an 'exact' solution by means of the geometric methods of our paper, and only perturbative techniques or numerical analysis would allow us to do this computation.

Quantization and reduction. In this paper we did not touch the issue of symmetry reduction. In particular, it would be interesting to check if the Guillemin–Sternberg conjecture [20] (see also [18]) holds in the case of a free rigid body; this question was posed to us by J Marsden. (We recall that the Guillemin–Sternberg conjecture states the commutation between the reduction and the quantization procedures.) In fact, the group $SO(3)$ acts as a group of symmetries on a free rigid body. A cosymplectic reduction procedure (analogous to the Marsden–Weinstein reduction procedure) could be formulated. A similar analysis has been carried out in [31] in order to formulate a geometric prequantization (see also [48] for similar results under stronger hypotheses). The coadjoint orbits of constant angular momentum turn out to be spheres S^2. It would be very interesting to investigate the interplay between the two inequivalent quantum structures of the rigid body and the possible quantum structures of coadjoint orbits, especially in view of the fact that their topologies are different. But this will be the subject of future work.

Spin particles. As we already observed, in order to deal with many interesting physical situations it would be desirable to extend our model to the case of n spin particles. Covariant quantum mechanics of one spin particle already exists [8]. An extension to n spin particles would allow us, e.g., to deal with the anomalous Zeeman effect for a rigid body with spin rotating in a constant magnetic field. This problem cannot be dealt with here for time and space constraints, and is another research theme for the future.

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